UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

NON-LINEAR DIFFERENTIAL EQUATIONS HAVING QUADRATIC STIFFNESS TERMS

by

A.W. Babister, M.A., Ph.D.

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SUMMARY

The nature of solutions of the autonomous equation

$$\ddot{x} + (b_0 + b_1 x)\, \dot{x} + c_1 x + c_2 x^2 = 0$$

is considered. The trajectories in the $(x, \dot{x})$ phase plane are given for all combinations of signs of the b and c parameters. It is shown that self-sustaining oscillations may occur provided that both $b_0 = 0$ and $b_1 c_2 = 0$, and integral curves are given for the variation of x with t in such cases.

The stability of systems satisfying this differential equation is discussed, together with methods of improving the character of the vibrations.

LIST OF CONTENTS

General Introduction

This report is part of a general investigation of dynamical systems with non-linear characteristics. In an earlier report (Babister, 1972), certain ordinary second order non-linear differential equations were classified according to their methods of solution. In this report we consider the nature of solutions of differential equations of the form

$$\frac{d^2x}{dt^2} + f(x)\,\frac{dx}{dt} + g(x) = 0 \qquad (1)$$

Here x can be thought of as the displacement at time t. We shall consider both the variation of x with t (the integral curve), paying particular attention to its periodicity and boundedness, and also the trajectories of the system in the $(x, \dot{x})$ phase plane.

A number of very general theorems exist concerning the properties of autonomous systems such as that defined by (1), in which the functions f and g (relating to the damping and stiffness) depend on x only (see Sansone and Conti, 1964, and Minorsky, 1962). However, so many textbooks confine their attention to a very small number of equations of the type (1), notably to Van der Pol's equation, and the detailed solution of many other types is almost totally neglected.

In this report we consider equation (1) for the cases

$$f(x) = b_0 + b_1 x \tag{2}$$

and

$$g(x) = c_1 x + c_2 x^2 \tag{3}$$

where b_0, b_1, c_1 and c_2 are constants. Equation (1) then relates to the free vibration of a system with a quadratic restoring force and a non-linear damping force. Relaxation oscillations of Van der Pol's type (with limit cycles) do not occur in these cases, since f is a linear function of x and will thus be of different sign for large positive and large negative values of x (unless $b_1 = 0$). However, it will be shown that self-sustaining oscillations can occur for certain values of the b and c parameters. Physical examples of these oscillations (or closely related ones) occur in the analysis of the transient response of large amplitude motions of aircraft (Shinbrot, 1954) and in oscillations in surge chambers (Cole, 1927).

Before proceeding further with our particular piece of analysis, it is worthwhile to note some simple properties and transformations of equation (1). Firstly, since f and g are independent of t, it is obvious that if $x = \phi(t)$

is a solution of (1) then $x = \phi(t + c)$ is also a solution, where c is any constant. The complete solution of (1) will involve two arbitrary constants (which could correspond to the amplitude and phase). Thus, if the initial values of x and $\dot{x}$ are specified, the complete solution of (1) is uniquely determined provided that $f(x)$ and $g(x)$ have unique finite values for every value of x.

The physical characteristics of systems satisfying (1) can be seen by considering the function E defined by

$$E = \frac{1}{2}\dot{x}^2 + \int g(x)\,dx \;, \tag{4}$$

Differentiating (4) with respect to t, and using (1), we obtain

$$\frac{dE}{dt} = - f(x)\left(\frac{dx}{dt}\right)^2 \;. \tag{5}$$

We see that E may be regarded as the total energy of the system per unit mass; the dissipation of energy with time is proportional to (and has the same sign as) $f(x)$.

In equation (1) put

$$y = \dot{x} \tag{6}$$

Then (1) becomes

$$y \frac{dy}{dx} + f(x)y + g(x) = 0 \tag{7}$$

which is a first order equation relating x and $y (= \dot{x})$.
From (7),

$$\left.\begin{aligned} \frac{dy}{dx} &= - f(x) - \frac{g(x)}{y} \\ &\text{and} \\ \frac{d^2y}{dx^2} &= - f'(x) - \frac{g'(x)}{y} - \frac{f(x)g(x)}{y^2} - \frac{[g(x)]^2}{y^3}. \end{aligned}\right\} \tag{8}$$

These equations are of great help in drawing trajectories in the (x,y) phase plane. From (8) we see that $\frac{dy}{dx}$ will be infinite where $y = 0$ (unless $g(x) = 0$ at the same point in the phase plane).

To find possible slopes of trajectories passing through O, put $y = \lambda x$ in (8). On letting $x \to 0$, with $g(0) = 0$, we obtain

$$\lambda = - f(0) - \frac{g'(0)}{\lambda} ,$$

that is, using (2) and (3),

$$\lambda^2 + b_0 \lambda + c_1 = 0.$$

We see that, for the systems we are considering, the local slopes of trajectories passing through 0 satisfy the same characteristic equation as for the system obtained by retaining only the linear terms in (1).

Any periodic solution of (1) is represented by a closed curve in the phase plane, and this curve will intersect the x axis twice (at points corresponding to the maximum and minimum values of x on the closed curve). We note, too, that since x and y are related by eq. (6), if a trajectory in the phase plane does not cut the x axis, it must be an open trajectory, which starts at a point at infinity at $t = -\infty$ and ends at a point at infinity at $t = +\infty$. It is important to investigate the trajectories in all parts of the phase plane (and thus to consider the stability of any solution in relation to perturbations or errors in the initial data). The time variation along a trajectory can be deduced from (6); in particular, in the first and second quadrants of the phase plane, t increases with x, and in the third and fourth quadrants t increases as x decreases.

In (7) put $y = 1/z$. Then (7) becomes

$$\frac{dz}{dx} = f(x)z^2 + g(x)z^3 \tag{9}$$

This is a particular case of an Abel equation of the first kind.

For the system considered,

$$\frac{d^2x}{dt^2} + (b_0 + b_1x)\frac{dx}{dt} + c_1x + c_2x^2 = 0, \tag{10}$$

the point $x = 0$ is an equilibrium point. For such a system, the origin is a singular point in the $(x, \dot{x})$ phase plane. It is also seen that the point $x = -c_1/c_2$ is another equilibrium point for the system. On putting $x = z - c_1/c_2$ we see that (10) becomes

$$\ddot{z} + (b_0 - \frac{b_1c_1}{c_2} + b_1z)\dot{z} - c_1z + c_2z^2 = 0, \tag{11}$$

which is an equation of the same form as (10). More generally, any equation of the form

$$\ddot{z} + (b_0 + b_1 z)\,\dot{z} + c_0 + c_1 z + c_2 z^2 = 0 \tag{12}$$

in which c_1 and c_2 are not both zero, can be put in the form of (10) with real coefficients on letting $x = z - \gamma$ where γ is a real constant, provided that the equation

$$c_0 + c_1\gamma + c_2\gamma^2 = 0$$

has a real root, i.e. provided that $c_1^2 \geqslant 4c_0c_2$.

In (10), put $x = \alpha X$, $t = \beta T$ where α and β are constants. Then

$$\frac{d^2X}{dT^2} + (\beta b_0 + \alpha\beta b_1 X)\,\frac{dX}{dT} + \beta^2 c_1 X + \alpha\beta^2 c_2 X^2 = 0 \tag{13}$$

Thus, if (10) has the solution $x = \phi\,(t)$, with $x = x_0$, $\dot{x} = y_0$ at $t = 0$, (13) has the solution $X = \alpha^{-1}\,\phi\,(\beta T)$, with $X = x_0/\alpha$ and $dX/dT = \beta y_0/\alpha$ at $T = 0$. In particular we note that a variation in the value of α merely affects the non-linear terms in (13), and that if β is replaced by $-\beta$, t is changed to $-t$. Thus, if (10) has a periodic solution, (13) with any non-zero α and β will also have a periodic solution. It follows that the properties of a number of different physical systems (having, for example, different periods and damping times) can be deduced from one configuration in the phase plane. If $\alpha = -1$ and $\beta = 1$, the coefficients b_1 and c_2 in (10) become $-b_1$ and $-c_2$, and the variation of X with T is identical with that of $(-x)$ with t. Again, if $\alpha = 1$ and $\beta = -1$, the coefficients b_0 and b_1 in (10) become $-b_0$ and $-b_1$, and the variation of X with T is identical with that of x with $(-t)$. Thus the positive semi-trajectory $(T>0)$ in the X plane is the same as the negative semi-trajectory $(t<0)$ in the x plane.

These scaling factors α and β are of considerable importance in enabling us to relate solutions of (10) and (13). As will be shown in this report, they enable us to reduce drastically the number of combinations of the parameters b and c to be considered in order to determine the nature of the solutions of (10). The ordinates and abscissae of the phase plane figures are (wherever possible) chosen so that they do not alter when scaling factors α and β are applied; the ordinates and abscissae are, in such cases, non-dimensional parameters, if x has the dimensions of length.

Scaling factors were used in the numerical solutions given in this report, many of which were calculated on Glasgow University's analogue computer (PACE). (In this connection the author records his thanks to Professor T.R.F. Nonweiler for help in programming and operating the computer). The computer calculations were carried out for the equation

$$0.1\frac{d^2X}{dT^2} + (0.1b_0 + 0.2b_1X)\frac{dX}{dT} + 0.1c_1X + 0.2c_2X^2 = 0 \qquad (14)$$

with b_0, b_1, c_1 and c_2 each having the values 1, 0, -1. Thus the solutions were performed in real time ($\beta = 1$) with a scaling factor $\alpha = 2$. The magnitudes of the initial conditions were then never greater than unity.

In part 1 of this report we discuss the nature of solutions of the differential equation (10) with $c_2 = 0$, and in part 2 we deal with the case $c_2 \neq 0$. For ease of reference the various cases considered are set out in table 1.

TABLE 1

Index to discussion of solutions of

$$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_1 x + c_2 x^2 = 0$$

Para.	Case	b_1	c_1	c_2	General Remarks
			PART ONE		
1.2	1	0	0	0	
1.2	2	+	0	0	
1.2	3	-	0	0	
1.3	4	0	+	0	
1.4	5	+	+	0	Some periodic solutions
1.4	6	-	+	0	
1.5	7	0	-	0	
1.6	8	+	-	0	
1.6	9	-	-	0	
			PART TWO		
2.2	10	0	0	+	
2.2	11	+	0	+	
2.2	12	-	0	+	
2.3	13	0	0	-	
2.3	14	+	0	-	
2.3	15	-	0	-	
2.4	16	0	+	+	Some periodic solutions
2.4	17	+	+	+	
2.4	18	-	+	+	
2.5	19	0	+	-	Some periodic solutions
2.5	20	+	+	-	
2.5	21	-	+	-	
2.6	22	0	-	+	Some periodic solutions
2.6	23	+	-	+	
2.6	24	-	-	+	
2.6	25	0	-	-	Some periodic solutions
2.6	26	+	-	-	
2.6	27	-	-	-	

PART 1

Non-linear differential equations of the form

$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_1 x = 0$

1.1 Introduction

We consider solutions of the equation

$$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_1 x = 0 \tag{15}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b_0 + b_1 x)y - c_1 x \end{aligned}\right\} \tag{16}$$

where b_0, b_1 and c_1 are real constants. In particular we shall show how the nature of the solution depends upon the initial conditions $x = x_0$ and $y = \dot{x}_0 = y_0$ at time $t = 0$. We know that, for the given equation, the solution is uniquely determined once the initial conditions are specified.

1.2 Systems with zero stiffness ($c_1 = 0$)

If $c_1 = 0$, equation (15) is of the form

$$\ddot{x} + f(x)\dot{x} = 0 \tag{17}$$

Equation (17) has as its first integral

$$\dot{x} + F(x) = C \tag{18}$$

where

$$F(x) = \int f(x)dx$$

and C is a constant.

From (18) we obtain

$$t + B = \int \frac{dx}{C - F(x)}$$

where B is a constant. We see that exceptional (point) solutions occur for $F(x) = C$. If $F(x)$ is a continuous function, this means that the set of point solutions $(x = \text{const.}, y = \dot{x} = 0)$ is dense on at least part of the x axis. These points correspond to equilibrium positions of the system (17).

<u>Case 1</u> $b_1 = 0, c_1 = 0$

$$\ddot{x} + b_0\dot{x} = 0 \tag{19}$$

Equation (19) is a linear differential equation, its general solution being (for $b_0 \neq 0$)

$$\left.\begin{aligned} x &= Ae^{-b_0 t} + B, \\ y &= \dot{x} = -Ab_0 e^{-b_0 t} = Bb_0 - b_0 x, \end{aligned}\right\} \tag{20}$$

where A and B are constants. If $b_0 = 0$, the solution of (19) is

$$\left.\begin{aligned} x &= At + B \\ y &= \dot{x} = A. \end{aligned}\right\} \qquad (21)$$

From (20) and (21) we see that the trajectories in the phase plane are linear (see fig. 1, in which y/b_0 is plotted against x for $b_0 > 0$). If $y_0 = 0$, then $x = B$ for all values of t and the phase plane curve is a point (a singular point). Thus we have a whole line of singular points (points of equilibrium for the system) along the x axis. From (20) we see that, if $b_0 > 0$, the solution is bounded as $t \to \infty$; the motion is a subsidence, $x \to B$ as $t \to \infty$. If $b_0 \leq 0$, the solution is unbounded as $t \to \infty$; the motion is a divergence, $x \to \pm\infty$ as $t \to \infty$ (unless $y_0 = 0$). Figure 1 is drawn for $b_0 > 0$. If $b_0 < 0$, the arrows on the curves should be reversed (this, as shown in the general introduction, corresponds to a change in sign of β, the time scaling factor).

<u>Case 2</u> $b_1 > 0$, $c_1 = 0$.

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} = 0. \qquad (22)$$

For simplicity we first consider the equation

$$\ddot{x} + b_1 x\dot{x} = 0, \quad \text{with} \quad b_1 > 0. \qquad (23)$$

The general solution of (23) can be written in the following forms:-

$$\left.\begin{aligned} &\text{if } y_0 \geqslant 0, \quad x = A \tanh\left(\tfrac{1}{2}Ab_1 t + B\right) \\ &\qquad y = \dot{x} = \tfrac{1}{2}A^2 b_1 \operatorname{sech}^2\left(\tfrac{1}{2}Ab_1 t + B\right) = \tfrac{1}{2}b_1\left(A^2 - x^2\right) \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} &\text{if } y_0 + \tfrac{1}{2}b_1 x_0^2 > 0 > y_0 \quad x = A \coth\left(\tfrac{1}{2}Ab_1 t + B\right) \\ &\qquad y = \dot{x} = -\tfrac{1}{2}A^2 b_1 \operatorname{cosech}^2\left(\tfrac{1}{2}Ab_1 t + B\right) = \tfrac{1}{2}b_1\left(A^2 - x^2\right), \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} &\text{if } y_0 + \tfrac{1}{2}b_1 x_0^2 = 0, \quad x = \left(\tfrac{1}{2}b_1 t + B\right)^{-1} \\ &\qquad y = \dot{x} = -\tfrac{1}{2}b_1\left(\tfrac{1}{2}b_1 t + B\right)^{-2} = -\tfrac{1}{2}b_1 x^2, \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} &\text{if } y_0 + \tfrac{1}{2}b_1 x_0^2 < 0, \quad x = -a \tan\left(\tfrac{1}{2}ab_1 t + B\right) \\ &\qquad y = \dot{x} = -\tfrac{1}{2}a^2 b_1 \sec^2\left(\tfrac{1}{2}ab_1 t + B\right) = -\tfrac{1}{2}b_1\left(a^2 + x^2\right), \end{aligned}\right\} \quad (27)$$

where a, A and B are constants (determined from the initial conditions).

From (24) - (27), we see that the trajectories in the phase plane are arcs of parabolas (see fig. 2, in which y/b_1 is plotted against x for $b_0 = 0$). If $y_0 = 0$, B is infinite and $x = A$ for all values of t; the phase plane curve is then a point. Thus (as in case 1) we have a whole line of equilibrium points along the x axis. From (24) - (26) we see that, if $b_1 > 0$, the solution is bounded and $x \to A$ (or zero) as $t \to \infty$ provided that either $y_0 \geq 0$ or $x_0 \geq (-2y_0/b_1)^{1/2} \geq 0$; if neither of these conditions is satisfied, $x \to -\infty$ either as $t \to \infty$ (from equation (25))or at a finite time (from equations (26) and (27)), as shown by the curves in the third quadrant of the phase plane of figure 2. As in case 1, along a given trajectory $y(= \dot{x})$ does not change sign.

Equation (22) can be reduced to a equation of the form of (23) on putting $z = x + b_0/b_1$, and the corresponding general solutions follow from (24) - (27). The trajectories in the phase plane are still arcs of parabolas, being given by

$$y + \frac{1}{2b_1}(b_0 + b_1 x)^2 = \text{const.} \tag{28}$$

It is seen that the general configuration in the phase plane is identical with that of figure 2, apart from a change of origin.

<u>Case 3</u> $b_1 < 0,\ c_1 = 0$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} = 0 \tag{29}$$

On putting $x = -X$, (29) becomes

$$\ddot{X} + (b_0 - b_1 X)\dot{X} = 0 \tag{30}$$

which is of the same form as (22). Thus the integral curves in this case are found by applying a scaling factor $\alpha = -1$ to those of case 2. The trajectories in the phase plane are given by (28), the point $(x, y/|b_1|)$ in figure 2 being transformed into the point $(-x, -y/|b_1|)$. We see that, if $b_1 < 0$, the solution is bounded as $t \to \infty$ provided that either $y_0 \leqslant 0$ or $x_0 \leqslant -(2y_0/|b_1|)^{1/2} \leqslant 0$.

1.3 Systems with linear damping and positive stiffness $(b_1 = 0, c_1 > 0)$

Case 4

$$\ddot{x} + b_0\dot{x} + c_1 x = 0, \quad c_1 > 0. \tag{31}$$

Equation (31) is a linear differential equation, its general solution being:-

$$\left.\begin{aligned} \text{if } b_0^2 > 4c_1, \quad x &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ y = \dot{x} &= \lambda_1 Ae^{\lambda_1 t} + \lambda_2 Be^{\lambda_2 t} \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} &\text{if } b_0^2 = 4c_1, \quad x = Ae^{-b_0 t/2} + Bte^{-b_0 t/2} \\ &y = \dot{x} = (B - \tfrac{1}{2}b_0 A)e^{-b_0 t/2} - \tfrac{1}{2}b_0 Bte^{-b_0 t/2} \end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned} &\text{if } b_0^2 < 4c_1, \quad x = Ae^{-b_0 t/2} \sin(\mu t + B) \\ &y = \dot{x} = \sqrt{c_1}\, Ae^{-b_0 t/2} \cos(\mu t + B + \nu), \end{aligned}\right\} \quad (34)$$

where A and B are constants determined from the initial conditions. Here λ_1 and λ_2 are the real roots of the characteristic equation

$$\lambda^2 + b_0\lambda + c_1 = 0, \quad (35)$$

$$\left.\begin{aligned} &\mu = +\sqrt{c_1 - \tfrac{1}{4} b_0^2} \\ &\sin \nu = b_0/2\sqrt{c_1} \\ \text{and} \quad &\cos \nu = \mu/\sqrt{c_1} \end{aligned}\right\} \quad (36)$$

From (4) and (31),

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}c_1 x^2 \quad (37)$$

and

$$\frac{dE}{dt} = - b_0 \dot{x}^2 \tag{38}$$

Thus, if $b_0 = 0$, the total energy E of the system remains constant, and the trajectories in the phase plane are closed curves (concentric ellipses) around the point 0 showing (as is well known) that the motion is periodic, the period being $2\pi / \sqrt{c_1}$. The origin 0 is the only singular point in the phase plane (it corresponds to the only equilibrium position in this case); it is called a center.

If $b_0 > 0$, E decreases with time, and from (34) we see that the trajectories in the phase plane are spiral curves around 0 (provided that $b_0^2 < 4c_1$), the point 0 being reached at $t = + \infty$ (see figure 3 in which $y/ \sqrt{c_1}$ is plotted against x for $b_0/ \sqrt{c_1} = 1$). The motion is then a damped oscillation, and 0 is a stable focus in the phase plane. Figure 4 shows the effect of the damping parameter b_0 on the trajectory for the initial conditions $x_0 = 1$, $y_0 = 0$ (i.e, initial position displacement but zero velocity); precisely similar curves will result for any other initial amplitude (with $y_0 = 0$), since (31) is a linear differential equation and, if $x = \phi (t)$ is a solution, so is $x = C\phi (t)$ where C is any constant.

If $b_0 > 2\sqrt{c_1}$, we see from (32) that the motion is composed of two modes, both of which are subsidences. Figure 5 shows the trajectories for $b_0/\sqrt{c_1} = 2.5$. There are two linear trajectories $y = \lambda_1 x$, $y = \lambda_2 x$ (in this example $\lambda_1 = -0.5\sqrt{c_1}$, $\lambda_2 = -2.0\sqrt{c_1}$) corresponding to the values $A \neq 0$, $B = 0$ and $A = 0$, $B \neq 0$ in (32). The point 0 is said to be a stable two-tangent node; the trajectory $A = 0$ has slope λ_2 (the numerically larger root) while all other trajectories have slope λ_1 at 0. The solutions $\dot{x} = \lambda_1 x$ and $\dot{x} = \lambda_2 x$ are examples of particular solutions of the linear equation (31). We shall see that it is of importance to find if there are any particular solutions of non-linear equations (for which a general solution may not exist). If $b_0 = 2\sqrt{c_1}$, all the trajectories have a common tangent $y = -\frac{1}{2} b_0 x$ at the origin, and 0 is called a stable one-tangent node.

If $b_0 < 0$, from (38), the total energy increases with time, and from (32) - (34) $E \rightarrow \infty$ as $t \rightarrow \infty$, the motion being an increasing oscillation if $b_0^2 < 4c_1$; in that case, as shown in figure 4, the trajectories spiral away from 0 and the origin is an unstable focus. As shown in the general introduction, a change of sign of b_0 (with c_1 remaining constant) is equivalent to having a time scaling factor $\beta = -1$. The configuration in the phase plane is unaltered, but the arrows on the curves should be reversed. In accordance with the terminology of Sansone and Conti, for the system (31), the solution $x = 0$, $y = 0$ is asymptotically stable for $t \to +\infty$ if $b_0 > 0$ and is unstable if $b_0 < 0$. If $b_0 = 0$, 0 is a center and the solution $x = 0$, $y = 0$ is said to be stable (not asymptotically). We note that changing the sign of the time scaling factor β does not change the character of a center, but changes a stable point into an unstable one and vice versa.

1.4 Systems with non-linear damping and positive stiffness

Case 5 $b_1 > 0$, $c_1 > 0$.

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x = 0. \tag{39}$$

We first consider the equation

$$\ddot{x} + b_1 x\dot{x} + c_1 x = 0, \quad \text{with } b_1 > 0,\ c_1 > 0. \tag{40}$$

This is a particular example of the class of differential equations of the form

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0 \tag{41}$$

in which $f(x)$ and $g(x)$ are odd functions of x, with

$g(x) > 0$ for $x > 0$. Then, with $F(x) = \int_0^x f(s)ds$, if there exists a positive number a such that

$$\int_0^x \frac{g(s)}{F(s)}\, ds > \frac{1}{4}\left| F(x) \right| \tag{42}$$

for $0 < x \leq a$, then all the trajectories of (41) in the neighbourhood of the origin ($x = 0$, $\dot{x} = 0$) are closed (see Opial (1958) and Sansone and Conti (1964)). For equation (40), the inequality (42) holds for all positive a (the l.h.s. of (42) is infinite in this case), and as we shall see there is a large part of the phase plane for which the trajectories are closed, and thus self-sustaining ocsillations are possible.

The general first integral of (40) is

$$\tfrac{1}{2}b_1^2x^2 + b_1y - c_1 \log \left| b_1y + c_1 \right| = \text{const.}, \tag{43}$$

where $$y = \dot{x} .$$

We note that (40) has the particular first integral $y = \dot{x} = - c_1/b_1$.

From (43) we see that the form of the trajectories will be very different according to whether $y >$ or $<$ $(-c_1/b_1)$. In fig. 6, if $b_1 y_0/c_1 > -1$, the trajectories are closed curves, enclosing 0 and symmetrical w.r.t. Oy. There is, in fact, a periodic solution of (40) for any system having $x_0 = 0$, $y_0 > 0$ at $t = 0$. The variation of x with t (as determined by analogue computer) is shown in fig. 7 for various values of $b_1 y_0/c_1$. We see that the period varies little with y_0 (for the range of values considered). An approximate formula for the period can be found by writing the solution of (40) in the form

$$x = X_0 + b_1 X_1 + b_1^2 X_2 + \ldots.$$

and

$$c_1 = \alpha_0 + b_1 \alpha_1 + b_1^2 \alpha_2 + \ldots.,$$

b_1 being considered as the perturbation parameter. This was the method used by Poincaré for the solution of perturbation problems in celestial mechanics (see also McLachlan, 1950). We find the period is (to the second order in b_1)

$$\frac{2\pi}{\sqrt{c_1}} \left(1 + \frac{y_0^2 b_1^2}{24 c_1^2} \right),$$

which again shows the small effect the non-linear term has. If $b_1 y_0/c_1 \leq -1$, the trajectories are open curves, the displacement tending to infinity as $t \to +\infty$.

Figure 8 shows the trajectories for the system (39) if $b_0 > 0$ (there is then no exact first integral). We see that, for much of the phase plane, the trajectories spiral in to 0. This is to be expected since, for this system, from (5),

$$\frac{dE}{dt} = -(b_0 + b_1 x)\dot{x}^2$$

and thus the energy will be decreasing if $b_0 + b_1 x > 0$. In the neighbourhood of 0 the effect of the non-linear term in (39) is very small and the trajectories there resemble those of the damped linear system (case 4, fig. 3). However, below the line AB, in part of the region $b_1 y/c_1 < -1$, non-linear effects predominate and, as above, the trajectories are open. The curve AB is a separatrix for the system (39); it separates the domain in which trajectories spiral in to 0 from that in which they go off to infinity. In fig. 6, the line $y = -c_1/b_1$ is another example of a separatrix.

The trajectories of the system (39) with $b_0 < 0$ can be obtained (as shown in the general introduction) by employing scaling factors $\alpha = -1$, $\beta = -1$ (this changes the sign of b_0 but b_1 and c_1 are unaltered). Thus the point (x,y) in figure 8 is transformed into the point $(-x,y)$; in addition the arrows on the curves should be reversed. The phase plane diagram for $b_0 < 0$ becomes the mirrorimage of that of fig. 8 with respect to the y axis, the trajectories spiralling away from 0. We see that, if $b_0 < 0$, all the trajectories tend to infinity as $t \to \infty$.

Case 6 $\quad b_1 < 0, \quad c_1 > 0, \quad c_2 = 0$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x = 0. \tag{44}$$

On putting $x = -X$, (44) becomes

$$\ddot{X} + (b_0 - b_1 X)\,\dot{X} + c_1 X = 0 \tag{45}$$

which is of the same form as (39). Thus the integral curves can be found by applying a scaling factor $\alpha = -1$ to those of case 5. Here too we see that, if $b_0 = 0$, self-sustaining oscillations around 0 can occur if $|b_1|\, y/c_1 < 1$, 0 being a center for the system (44). If $b_0 > 0$, the trajectories will, in general, spiral in towards 0 (unless y_0 is large, i.e, on the other side of the separatrix); if $b_0 < 0$, all the trajectories tend to infinity as $t \to \infty$.

The oscillatory motion we have just considered in cases 5 and 6 also occurs in systems with quadratic damping. Thus the system

$$\ddot{z} + \tfrac{1}{2}b_1\dot{z}^2 + c_1 z = 0 \tag{46}$$

becomes identical with (40) on differentiating with respect to t and putting $x = \dot{z}$. Physical systems with quadratic damping (such as turbulence damping) satisfy equations of the type

$$\ddot{z} + \tfrac{1}{2}b_1 \left| \dot{z} \right| \dot{z} + c_1 z = 0 \tag{47}$$

or the equivalent form

$$\ddot{x} + b_1 \left| x \right| \dot{x} + c_1 x = 0 \tag{48}$$

For systems such as (47) or (48), the trajectories for the two halves of the oscillation in which x has the same sign are identical with those of cases 5 and 6 which we have just considered (but with different constants of integration). However, from (48), the coefficient of $\dot{x}$ is never negative (with $b_1 > 0$), and the composite trajectory tends to 0 as $t \to \infty$.

Both linear and quadratic damping are used in the Lewis servomechanism (Lewis, 1952, and Caldwell and Rideout, 1953), which satisfies the equation

$$\ddot{x} + (b_0 + b_1 \,|x|\,)\,\dot{x} + c_1 x = 0. \tag{49}$$

Here again the curves for x positive and x negative can be deduced from fig. 8. The idea behind this servomechanism was to ensure a more rapid response to large errors than in the corresponding linear system.

1.5 <u>Systems with linear damping and negative stiffness</u> $(b_1 = 0,\ c_1 < 0)$

<u>Case 7</u> $\quad \ddot{x} + b_0\dot{x} + c_1 x = 0, \quad c_1 < 0 \qquad (50)$

Equation (50) is a linear differential equation, its general solution being:-

$$\left.\begin{aligned} x &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ y &= \dot{x} = \lambda_1 Ae^{\lambda_1 t} + \lambda_2 Be^{\lambda_2 t} \end{aligned}\right\} \tag{51}$$

where λ_1 and λ_2 are real roots of the characteristic equation

$$\lambda^2 + b_0\lambda + c_1 = 0. \tag{52}$$

Since c_1 is negative, one root λ_1 will be positive (corresponding to a divergence) and the other root λ_2 will be negative (corresponding to a subsidence).

If $b_0 = 0$, (50) has the first integral

$$\frac{1}{2}y^2 + \frac{1}{2}c_1x^2 = 0 \tag{53}$$

where

$$y = \dot{x}.$$

Thus, with $c_1 < 0$, the trajectories in the phase plane are branches of rectangular hyperbolas (as shown in fig. 9), the two linear trajectories $y/\sqrt{-c_1} = \pm x$ corresponding to zero values of A or B in (51). We see that, in general, the trajectories tend to infinity as $t \to \infty$. 0 is a saddle point (this corresponds to an unstable position of equilibrium).

If $b_0 > 0$, the trajectories in the phase plane are as in fig. 10 (which is drawn for $b_0/\sqrt{-c_1} = 1$). Here too 0 is a saddle point. The linear trajectories correspond to $y = \lambda_1 x$, $y = \lambda_2 x$. Trajectories for $b_0 < 0$ can be obtained from those in fig. 10 by having a time scaling factor $\beta = -1$; thus the point (x,y) in fig. 10 is transformed into the point $(x,-y)$. Thus (as is well known) all linear systems (50) having negative stiffness diverge to infinity as $t \to \infty$ (except for the principal mode corresponding to $y = \lambda_2 x$).

1.6 Systems with non-linear damping and negative stiffness

Case 8 $\quad b_1 > 0, \; c_1 < 0.$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x = 0. \tag{54}$$

We first consider the equation

$$\ddot{x} + b_1 x\dot{x} + c_1 x = 0, \quad \text{with} \quad b_1 > 0, \; c_1 < 0. \tag{55}$$

As shown in para. 1.4, the general first integral of (55) is

$$\tfrac{1}{2} b_1^2 x^2 + b_1 y - c_1 \log \left| b_1 y + c_1 \right| = \text{const.}, \tag{56}$$

where

$$y = \dot{x}.$$

However, due to the change in sign of c_1, the trajectories are very different from those of case 5. Eq. (55) also has the particular first integral $y = \dot{x} = - c_1/b_1$.

The trajectories for systems given by (55) and (54) are shown in figures 11 and 12 respectively. We see that these figures have much in common with figs. 9 and 10, in which the damping was linear. In particular 0 is a saddle point for the non-linear systems and all the trajectories tend to infinity as $t \to \infty$, except for the two trajectories which converge on the origin; these are two arms (or separatrices) of the singular point O. As can be seen from figs. 11 and 12, all the other trajectories tend to approach the other two arms of the singular point O as $t \to \infty$.

Trajectories for $b_0 < 0$ can be obtained from those in fig. 12 by having scaling factors $\alpha = -1$, $\beta = -1$. The point (x,y) in fig. 12 is transformed into the point $(-x,y)$, and in addition the arrows on the curves should be reversed. Once again we find that all the trajectories go to infinity as $t \to \infty$, except for two arms of the singular point O.

<u>Case 9</u> $b_1 < 0$, $c_1 < 0$.

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x = 0. \qquad (57)$$

On putting $x = -X$, (57) becomes

$$\ddot{X} + (b_0 - b_1 X)\,\dot{X} + c_1 X = 0 \qquad (58)$$

which is of the same form as (54). Thus the integral curves can be found by applying a scaling factor $\alpha = -1$ to those of case 8. All the trajectories tend to infinity as $t \to \infty$ with the exception of two arms of the singular point O.

PART 2

Non-linear differential equations of the form

$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0$

2.1 Introduction

We consider solutions of the equation

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0 \tag{59}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b_0 + b_1 x)\,y - c_1 x - c_2 x^2 \end{aligned}\right\} \tag{60}$$

where b_0, b_1, c_1 and c_2 are constants $(c_2 \neq 0)$. The point 0 is obviously an isolated singular point for the system (60); however, as shown below, the term involving c_2 may cause the system to have a non-elementary singular point at 0. As in part 1, we shall show how the nature of the solution depends upon the initial conditions $x = x_0$ and $y = \dot{x}_0 = y_0$ at time $t = 0$. In particular we shall determine whether or not there exist trajectories tending towards 0.

2.2 Systems with square-law stiffness $(c_1 = 0,\ c_2 > 0)$

Case 10 $\quad b_1 = 0,\ c_1 = 0,\ c_2 > 0.$

$$\ddot{x} + b_0\dot{x} + c_2 x^2 = 0. \tag{61}$$

We first consider the equation

$$\ddot{x} + c_2 x^2 = 0, \quad \text{with} \quad c_2 > 0. \tag{62}$$

The general first integral of (62) is

$$\frac{1}{2} y^2 + \frac{1}{3} c_2 x^3 = A \tag{63}$$

where $y = \dot{x}$ and A is a constant. The general solution for x in terms of t can be expressed as a Weierstrass elliptic function (Whittaker and Watson, 1927). We find

$$x = -(6/c_2)\ \wp\,(t + B; 0, -Ac_2^2/18) \tag{64}$$

where B is an arbitrary constant. However, the nature of the solution can most easily be seen in the (x,y) phase plane (fig. 13) in which $y/\sqrt{c_2}$ is plotted against x, for $b_0 = 0$. If $A = 0$, in (63), we see that the corresponding trajectories are given by

$$y = \pm\, (-2c_2 x^3/3)^{\frac{1}{2}} .$$

These two trajectories have the singular point 0 as a limit; one of them approaches 0 while the other recedes from 0, both trajectories being tangent to the x axis at 0. We see that 0 is a non-elementary singular point, and is obviously a position of unstable equilibrium. Physically, eq. (62) corresponds to a conservative system with its potential energy having a point of inflexion at $x = 0$. These two trajectories form the two arms of the separatrix through 0, the other trajectories curving round on either side of 0. We see that, in general, the trajectories tend to infinity as t increases. From (64) we find that the point at infinity may be reached in a finite time.

Fig. 14 shows the corresponding trajectories for the system (61) with $b_0 > 0$. In this case (61) has no general first integral. Here too 0 is a non-elementary singular point; as shown in the general introduction, the slopes λ of trajectories passing through 0 satisfy the equation

$$\lambda^2 + b_0\lambda = 0. \quad (\text{Since } c_1 = 0);$$

that is, $\lambda = 0$ or $-b_0$. This can also be seen from fig. 14. We see that, for systems with $x_0 = 0$ and $0 < c_2y_0/b_0^3 < 1.5$, the trajectories turn inwards to the origin, in a similar manner to that for a stable node (this is due to the damping term). There is also one trajectory $A0$ converging directly on 0; this is one arm of the separatrix through 0. All the other trajectories (including $0B$, the other arm of the separatrix) tend to infinity as t increases.

Fig. 14 is drawn for $b_0 > 0$. Trajectories for $b_0 < 0$ can be obtained from those in fig. 14 by having a time scaling factor $\beta = -1$. Thus the point (x,y) in fig. 14 is transformed into the point $(x,-y)$; in addition the arrows on the curves should be reversed.

We see that, if $b_0 < 0$, all the trajectories tend to infinity as t increases (except for that corresponding to BO).

Case 11 $\qquad b_1 > 0,\ c_1 = 0,\ c_2 > 0.$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_2 x^2 = 0 \tag{65}$$

We first consider the equation

$$\ddot{x} + b_1 x\dot{x} + c_2 x^2 = 0$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -b_1 xy - c_2 x^2 \end{aligned}\right\} \tag{66}$$

This is a particular case of the system

$$\dot{x} = y + f\,(x,y)$$

$$\dot{y} = g\,(x,y)$$

considered by Keil (1955), f and g being non-linear functions. Here too, O is a non-elementary singular point.

The trajectories in the phase plane for the system (66) are shown in fig. 15 (in which $b_1^3 y/c_2^2$ is plotted against $b_1^2 x/c_2$). The general form of the trajectories is very similar to that in fig. 13.

There is one trajectory which terminates at O; all other trajectories tend to infinity as t increases. Here too O is a position of unstable equilibrium.

Trajectories for the system (65), for $b_0 > 0$, are shown in fig. 16 (for $b_0b_1/c_2 = 1$). As in case 10, the slopes λ of trajectories passing through O are given by $\lambda = 0$ or $-b_0$. For $b_0b_1/c_2 = 1$, we see that two trajectories converging on O are given by

$$y = -b_0x.$$

This corresponds to the particular integral

$$\dot{x} = -b_0x \tag{67}$$

of (65) for this value of the ratio (b_0b_1/c_2). Putting $u = \dot{x} + b_0x$, we see that (65) becomes

$$\dot{u} + b_1ux = 0. \tag{68}$$

From fig. 16 we see that no trajectory crosses the line AOB, $y = -b_0x$. Trajectories to the right of this line (in the first, second and fourth quadrants) turn in towards O, having a common tangent along Ox at O. As can be seen from (68), if $x > 0$, trajectories on either side of AO approach this line as t increases, but never merge with it. We see that trajectories to the left of AOB (in the second, third and fourth quadrants) tend to infinity as t increases; one of these, OC, starts from O.

By considering the slopes of the trajectories of the system (65) with $b_0 > 0$, we find that, if $b_0 b_1/c_2 \leq 1$, most of the trajectories tend to infinity as t increases, only a few (for sufficiently small y_0 at $x_0 = 0$) turning in towards 0; the general picture is very similar to that given in fig. 14. If $b_0 > 0$ and $b_0 b_1/c_2 > 1$, some trajectories (in the first, second and fourth quadrants) turn in towards 0, while others (in the second, third and fourth quadrants) tend to infinity as t increases. An interesting example of this is for $b_0 b_1/c_2 = 2$, for which (65) has the particular integral

$$\dot{x} = -\tfrac{1}{2} b_1 x^2 \tag{69}$$

or, in the phase plane, $y = -\frac{1}{2} b_1 x^2$. Putting $u = \dot{x} + \frac{1}{2} b_1 x^2$, we see that (65) becomes (with $b_0 b_1/c_2 = 2$)

$$\dot{u} + b_0 u = 0,$$

which gives the first integral of (65) in the form

$$\dot{x} + \tfrac{1}{2} b_1 x^2 = C \exp(-b_0 t), \tag{70}$$

where C is an arbitrary constant. Thus, with $b_0 > 0$ and $b_0 b_1/c_2 = 2$, the trajectories tend to merge with the parabola $y = -\frac{1}{2} b_1 x^2$ as t increases.

There are no particular solutions if $b_0 < 0$ (with b_1 and c_2 both positive and $c_1 = 0$). A typical example of trajectories for the system (65) for $b_0 < 0$ is shown in fig. 17 (for $b_0 b_1 / c_2 = -1$). All the trajectories tend to infinity as t increases (except one trajectory which terminates at 0). As can be seen, there are some trajectories which commence at 0.

<u>Case 12</u> $\qquad b_1 < 0,\ c_1 = 0,\ c_2 > 0.$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_2 x^2 = 0. \qquad (71)$$

On putting $t = -T$, (71) becomes

$$\frac{d^2x}{dT^2} - (b_0 + b_1 x)\,\frac{dx}{dT} + c_2 x^2 = 0. \qquad (72)$$

Thus the integral curves can be found by applying a scaling factor $\beta = -1$ to those of case 11. The trajectories in the phase plane can be obtained from figures 15-17. We find that, in general, the trajectories tend to infinity as t increases, except that (i) if $b_0 > 0$, some trajectories close to the origin (in the first and fourth quadrants) terminate at 0 and (ii), there is always one arm of the separatrix which terminates at 0. Linear trajectories, corresponding to the particular integral (67) occur for $b_0 b_1 / c_2 = 1$ (with $b_0 < 0$).

2.3 Systems with square-law stiffness ($c_1 = 0,\ c_2 < 0$)

Case 13 $\qquad b_1 = 0,\ c_1 = 0,\ c_2 < 0.$

Case 14 $\qquad b_1 > 0,\ c_1 = 0,\ c_2 < 0.$

Case 15 $\qquad b_1 < 0,\ c_1 = 0,\ c_2 < 0.$

$$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_2 x^2 = 0. \tag{73}$$

On putting $x = -X$, (73) becomes

$$\ddot{X} + (b_0 - b_1 X)\dot{X} - c_2 X^2 = 0, \tag{74}$$

which is of the same form as (61), (65) or (71), depending on the sign of b_1. Thus the integral curves can be found by applying a scaling factor $\alpha = -1$ to those of cases 10, 12 and 11. The point (x,y) is transformed into the point $(-x,-y)$. We see that, in general, the trajectories tend to infinity as t increases, except that (i) if $b_0 > 0$, some trajectories close to the origin (in the second and third quadrants) terminate at O and (ii) there is always one arm of the separatrix which terminates at O. As in para. 2.2, linear trajectories occur for $b_0 b_1/c_2 = 1$.

2.4 Systems with quadratic stiffness ($c_1 > 0$, $c_2 > 0$)

As pointed out in the general introduction, the system

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0$$

has equilibrium points at both $x = 0$ and $x = -c_1/c_2$. Thus on putting $x = z - c_1/c_2$, we obtain the equation

$$\ddot{z} + (b_0 - \frac{b_1 c_1}{c_2} + b_1 z)\,\dot{z} - c_1 z + c_2 z^2 = 0. \qquad (75)$$

The phase plane thus has two singular points, but they are both elementary singular points of the type considered in part 1. This is seen by considering the form of the trajectories in the neighbourhood of each separate singular point; eq. (75) is more suitable for determining the nature of the singularity at $x = -c_1/c_2$ (for which $z = 0$). Thesetwo singularities coalesce if $c_2 = 0$, giving rise to thenon-elementary points at 0 mentioned in para. 2.2

Case 16 $\qquad b_1 = 0$, $c_1 > 0$, $c_2 > 0$.

$$\ddot{x} + b_0\dot{x} + c_1 x + c_2 x^2 = 0. \qquad (76)$$

We first consider the equation

$$\ddot{x} + c_1 x + c_2 x^2 = 0, \text{ with } c_1 > 0,\ c_2 > 0. \qquad (77)$$

The general first integral of (77) is

$$\frac{1}{2}y^2 + \frac{1}{2}c_1x^2 + \frac{1}{3}c_2x^3 = K, \tag{78}$$

where $y = \dot{x}$

and K is a constant. As in para. 2.2, the general solution for x in terms of t can be expressed as a Weirstrass elliptic function. We find

$$x = -(c_1/2c_2) - (6/c_2)\ \wp\ (t + B;\ c_1^2/12, \left[c_1^3 - 12Kc_2^2\right]/216) \tag{79}$$

where B is an arbitrary constant. The system (77) has no damping term and thus the total energy E is constant and equals K, from (78).

The trajectories in the phase plane are shown in fig. 18, in which $c_2y/c_1{}^{3/2}$ is plotted against c_2x/c_1, for $b_0 = 0$. We see that the system (77), with $b_1 = 0$ and $c_1 > 0$, does not have trajectories tending to O. The point O is a center and P, the point $(-c_1/c_2, 0)$, is a saddle point, the nature of the trajectories in the immediate neighbourhood of these two points being determined (in this case) by the linear terms in eq. (75) and (77). As pointed out by Sansone and Conti, the existence of a center cannot always be inferred by disregarding the non-linear terms. Loud (1964) has shown that a necessary condition that the system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 + \dots \end{aligned}\right\} \quad (80)$$

(where the omitted terms are of higher order) should have a center at 0 is that

$$AB + BC + E + 3G = 0. \quad (81)$$

If (81) does not hold, the system (80) has a focus at 0, which is stable or unstable according as the l.h.s. of (81) is negative or positive.

The variation of x with t (as determined by analogue computer) is shown in fig. 19 for various values of $c_2y_0/c_1^{3/2}$. If $c_2y_0/c_1^{3/2} < 1/\sqrt{3}$, the motion is periodic. The corresponding inequality for K, in (78), is $K < c_1^3/6c_2^2$. Using Poincaré's method, we find the period is

$$\frac{2\pi}{\sqrt{c_1}} \left(1 + \frac{5c_2^2y_0^2}{12c_1^3} \right),$$

to the second order in c_2. Thus the period increases with increase of amplitude. If $c_2y_0/c_1^{3/2} > 1/\sqrt{3}$, the motion is no longer periodic, x tending to $(-\infty)$ as t increases. This is shown in fig. 18, where the curve APBPC (the separatrix) divides the phase plane into three regions, the trajectories tending to infinity unless they are within the region PBP. On the arm of the separatrix, one trajectory AP starts from infinity and terminates at the singular point P, another PBP both starts and ends at P, and a third PC starts from P and goes to infinity as t increases.

Trajectories for the system (76) with $b_0 > 0$ are shown in fig. 20. The origin O is now a stable focus, the point P being still a saddle point. Here too, the nature of the trajectories near these two singularities is determined by the linear terms in eq. (75) and (76). The two arms of the separatrix APB divide the phase plane into two regions, in one of which all the trajectories spiral in to the origin, while in the other all the trajectories tend to infinity as t increases. Fig. 20 is drawn for $b_0/\sqrt{c_1} = 1$, and near O the trajectories closely resemble those for the linear system shown in fig. 3. The form of the trajectories for other positive values of b_0 does not differ remarkably from that shown in fig. 20; thus, for $b_0/\sqrt{c_1} = 2.5$, the trajectories in the region of O closely resemble those for the linear system shown in fig. 5.

Trajectories for $b_0 < 0$ can be obtained from those in fig. 20 by use of a time scaling factor $\beta = -1$. Thus the point (x,y) in fig. 20 is tranformed into the point $(x,-y)$; in addition the arrows on the curves should be reversed. We see that O is now an unstable focus and P is a saddle point; if $b_0 < 0$, all the trajectories tend to infinity as t increases (except those corresponding to OP and CP).

<u>Case 17</u> $b_1 > 0$, $c_1 > 0$, $c_2 > 0$.

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0. \qquad (82)$$

This is the most general case of (59) considered so far. As in case 16, the system (82) has two elementary singular points, at O and at $x = -c_1/c_2$, the latter point always being a saddle point in the phase plane for (82) with b_1, c_1 and c_2 all positive. The nature of the trajectories in the region of the origin depends upon the sign of b_0.

Figure 21 shows the trajectories for $b_0 = 0$. This case is of particular interest in that the corresponding linear equation has O as a center. However, in the present case, with <u>both</u> b_1 and c_2 non-zero, the necessary condition (81) for a center is not satisfied, and, if $b_0 = 0$ with $b_1 c_2 > 0$, O is an unstable focus. Thus in fig. 21, all the trajectories tend to infinity as t increases, except for the two arms of the separatrix which terminate at the saddle point. This figure is drawn for $b_1 \sqrt{c_1}/c_2 = 1$.

The phase plane diagram for $b_0 = 0$ and other positive values of b_1 does not differ remarkably from fig. 21.

Figures 22, 23 and 24 show the trajectories for (82) with $b_0/\sqrt{c_1} = 1, 2$ and -1 respectively. The origin is a stable focus (or stable node) for positive b_0 and an unstable one for negative b_0. This is precisely the same as for the corresponding linear system (shown in figs. 3-5). It will be seen from figs. 22 and 23 that, for the values of b_0 chosen, certain trajectories are straight lines. These trajectories correspond to particular solutions of (82). It is readily shown that (82) has a particular integral of the form

$$\dot{x} + b_0 x + c_1/b_1 = 0 \tag{83}$$

if $b_0 b_1/c_2 = 1$. This corresponds to the trajectory

$$y + \sqrt{c_1}\, x + c_1^{3/2}/c_2 = 0$$

in fig. 22. Again we find that (82) has a particular integral of the form

$$b_1 \dot{x} + c_2 x = 0 \tag{84}$$

if $b_0 b_1/c_2 = 1 + (b_1^2 c_1/c_2^2)$. This corresponds to the trajectory

$$y + \sqrt{c_1} x = 0$$

in fig. 23. In both of these figures we note that two arms of the separatrices through the saddle point divide the phase plane into a region in which the trajectories tend to 0 (mainly in the first, second and fourth quadrants) and a region in which the trajectories tend to infinity as t increases.

Equation (82) has a particular integral

$$(2c_2/b_1)\,\dot{x} + c_1 x + c_2 x^2 = 0 \qquad (85)$$

provided that $b_0 b_1/c_2 = 2 + (b_1^2 c_1/2c_2^2)$. (86)

The corresponding trajectory in the phase plane is

$$(2c_2/b_1)\,y + c_1 x + c_2 x^2 = 0. \qquad (87)$$

Putting $\quad u = (2c_2/b_1)\,\dot{x} + c_1 x + c_2 x^2,$

we see that (82) reduces to

$$b_1 u' + 2c_2 u = 0$$

provided that (86) holds true. This gives the first integral of (82) in the form

$$(2c_2/b_1)\,\dot{x} + c_1 x + c_2 x^2 = C \exp(-2c_2 t/b_1) \qquad (88)$$

where C is an arbitrary constant. Thus, with $c_2/b_1 > 0$ and b_0 satisfying (86), the trajectories tend to merge with the parabola, given by (87), as t increases. We note that this parabola passes through both the singular point at 0 and the one at $(-c_1/c_2, 0)$. If $b_1 > 0$, the phase plane portrait for these values of b_0 is not unlike that in fig. 23, the origin being a stable node and the point P a saddle point.

Case 18 $\quad b_1 < 0,\ c_1 > 0,\ c_2 > 0.$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0. \tag{89}$$

On putting $t = -T$, (89) becomes

$$\frac{d^2x}{dT^2} - (b_0 + b_1 x)\,\frac{dx}{dT} + c_1 x + c_2 x^2 = 0. \tag{90}$$

The integral curves can be found by applying a scaling factor $\beta = -1$ to those of case 17. The trajectories in the phase plane can be obtained from figures 21-24. The origin is a stable focus if $b_0 \geq 0$ and an unstable focus if $b_0 < 0$. The singularity at $(-c_1/c_2, 0)$ is a saddle point.

2.5 Systems with quadratic stiffness $(c_1 > 0,\ c_2 < 0)$

Case 19 $\quad b_1 = 0,\ c_1 > 0,\ c_2 < 0.$

Case 20 $\quad b_1 > 0,\ c_1 > 0,\ c_2 < 0.$

Case 21 $\quad b_1 < 0, \; c_1 > 0, \; c_2 < 0.$

$$\ddot{x} + (b_0 + b_1 x)\,\dot{x} + c_1 x + c_2 x^2 = 0. \tag{91}$$

On putting $x = -X$, (91) becomes

$$\ddot{X} + (b_0 - b_1 X)\,\dot{X} + c_1 X - c_2 X^2 = 0, \tag{92}$$

which is then of the same form as (76), (82) or (89) (depending on the sign of b_1). Thus the integral curves can be found by applying a scaling factor $\alpha = -1$ to those of cases 16, 18 and 17. The point (x,y) is transformed into the point $(-x, -y)$. We find that, if $b_1 = 0$, 0 is a center if $b_0 = 0$, a stable focus if $b_0 > 0$ and an unstable focus if $b_0 < 0$. If $b_1 > 0$, 0 is a stable focus if $b_0 \geq 0$ and an unstable focus if $b_0 < 0$. If $b_1 < 0$, 0 is a stable focus if $b_0 > 0$ and an unstable focus if $b_0 \leq 0$. The singularity at $(-c_1/c_2, 0)$ is a saddle point.

2.6 Systems with quadratic stiffness ($c_1 < 0$)

Case 22 $\quad b_1 = 0, \; c_1 < 0, \; c_2 > 0.$

Case 23 $\quad b_1 > 0, \; c_1 < 0, \; c_2 > 0.$

Case 24 $\quad b_1 < 0, \; c_1 < 0, \; c_2 > 0.$

<u>Case 25</u> $b_1 = 0,\ c_1 < 0,\ c_2 < 0.$

<u>Case 26</u> $b_1 > 0,\ c_1 < 0,\ c_2 < 0.$

<u>Case 27</u> $b_1 \leqslant 0,\ c_1 < 0,\ c_2 < 0.$

$$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_1 x + c_2 x^2 = 0. \quad (c_1 < 0) \tag{93}$$

In (93) put $x = z - c_1/c_2$. Then

$$\ddot{z} + \left(b_0 - \frac{b_1 c_1}{c_2} + b_1 z\right)\dot{z} - c_1 z + c_2 z^2 = 0. \quad (c_1 < 0) \tag{94}$$

We see that one effect of this transformation is to change the sign of the coefficient of x while leaving the coefficients of $x\dot{x}$ and x^2 unaltered. Thus the nature of solutions for cases 22-27 can be determined from those for cases 16-21, allowing for the change in the value of b_0, and for the displaced origin of the new phase diagram. We note, too, that cases 25, 26 and 27 can be related to cases 22, 23 and 24 respectively by applying a scaling factor $\alpha = -1$.

We find that, for cases 22-27, O is a saddle point, corresponding to a position of unstable equilibrium, as would be expected from a consideration of the linear terms in (93). Consider now the nature of the trajectories in the region of the other singular point P $(-c_1/c_2, 0)$. For cases 22 and 25, P is a center if $b_0 = 0$, a stable focus if $b_0 > 0$ and an unstable focus if $b_0 < 0$. For cases 23 and 27, P is a stable focus if $b_0 > b_1c_1/c_2$ and an unstable focus if $b_0 \leq b_1c_1/c_2$. For cases 24 and 26, P is a stable fo cus if $b_0 \geq b_1c_1/c_2$ and an unstable focus if $b_0 < b_1c_1/c_2$. Thus periodic solutions can occur in the neighbourhood of the point P only in cases 22 and 25 provided that $b_0 = 0$.

General Conclusions

We have been considering the transient behaviour of the dynamical system satisfying the equation

$$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_1 x + c_2 x^2 = 0. \tag{95}$$

For this system there is a unique solution (for given values of the parameters b and c) for initial conditions $x = x_0$, $\dot{x} = \dot{x}_0$ at $t = 0$. However, this solution may not exist for all positive t; thus in cases 2, 3, 10, 16 and 22, certain solutions tend to infinity at a finite time. This finite escape time cannot occur in physical systems (it can be closely approximated to with systems having very low stiffness, at least until saturation of some variable sets in).

The stability of the linear system

$$\ddot{x} + b_0\dot{x} + c_1 x = 0, \tag{96}$$

obtained by omitting the non-linear terms from (95), depends on the nature of the roots of the characteristic equation

$$\lambda^2 + b_0\lambda + c_1 = 0. \tag{97}$$

Such a system returns to its equilibrium position $(x = 0, \dot{x} = 0)$ if b_0 and c_1 are both positive; the equilibrium state is then said to be asymptotically stable. For the linear system this equilibrium state is asymptotically stable in the large (or globally stable) since every motion (however large) converges to the equilibrium state as $t \rightarrow \infty$.

Some of these results also apply to the non-linear system (95). As shown by Willems (1970), the null solution ($x = 0$, $\dot{x} = 0$) of the autonomous system (95) is asymptotically stable if all the roots of the characteristic equation (97) have negative real parts, that is, if b_0 and c_1 are both positive (as in figures 8, 20, 22 and 23). However, as shown in these figures, such systems are not globally asymptotically stable; indeed, one of the aims of this report is to give some quantitative data on the extent of the asymptotic stability, which is of great importane in practical applications. The null solution of the non-linear system (95) is unstable if at least one of the roots of (97) has a positive real part, that is, if $b_0 < 0$ or $c_1 < 0$; then 0 is either an unstable focus (or node) (as in figure 24) or a saddle point (in which case the null solution is said to be completely unstable, since all motions diverge from the neighbourhood of 0 as t increases).

As pointed out in para. 2.4, the stability of the system (95) cannot be inferred from that of the linear system (96) if the latter has a null solution with critical stability behaviour, that is, if $b_0 = 0$ and $c_1 > 0$. As stated in that paragraph, for such systems to have asymptotic stability it is necessary that $b_1c_2 < 0$ (cases 18 and 20).

So far we have considered the stability of the null state of the system (95). The system (95) also has the equilibrium state $x = -c_1/c_2$. As can be seen from para. 2.6 this equilibrium state is asymptotically stable if the roots of the equation

$$\lambda^2 + (b_0 - \frac{b_1c_1}{c_2})\lambda - c_1 = 0, \tag{98}$$

(the characteristic equation for the system (75)), have negative real parts, that is, if $b_0 > b_1c_1/c_2$ with $c_1 < 0$. It is unstable if $b_0 < b_1c_1/c_2$ or $c_1 > 0$ (as in figures 20-24).

The periodic solutions (cases 5, 6, 16, 19, 22 and 25) are orbitally stable (since a path which is sufficiently close to a periodic solution will always lie, in its entirety, in the immediate vicinity of the chosen orbit, provided the path does not coincide with a separatrix); for the system (95) these periodic solutions only occur provided that both $b_0 = 0$ and $b_1c_2 = 0$; and then only for trajectories sufficiently close to an equilibrium point. However, the period of rotation is different for the different paths, and thus the periodic solutions are not stable in the sense of Liapunov.

As can be seen from the phase plane mappings, the separatrices divide the phase plane into domains possessing very different properties. Thus (as in figure 23) on one side of the separatrix the trajectories may tend to an equilibrium point, whereas on the other side they may tend to infinity; the motion along any separatrix is asymptotic towards (or away from) a state of equilibrium.

One of the most important conclusions from this analysis is the very great change in the nature of the trajectories on the introduction of even a small non-linear term, as is seen by comparing figures 1 and 14, or 2 and 15. No limit cycles occur with the non-linear system (95); however, from a practical point of view, the undesireability of this system is that divergent motion will occur for large initial displacements from an equilibrium position. This is associated with the separatrices mentioned above. An improvement in the character of the vibrations of such a system would result by removing the separatrices as far as possible from the null point O, thus enlarging any region of stability. As can be seen from the phase plane diagrams, this can be done most easily by increasing c_1 (with b_0 and c_1 both positive) and decreasing (numerically) b_1 and c_2, thus decreasing the relative importance of the non-linear parameters. Such changes could come about either by direct alteration of the inertia and stiffness of the system or by using a feedback device.

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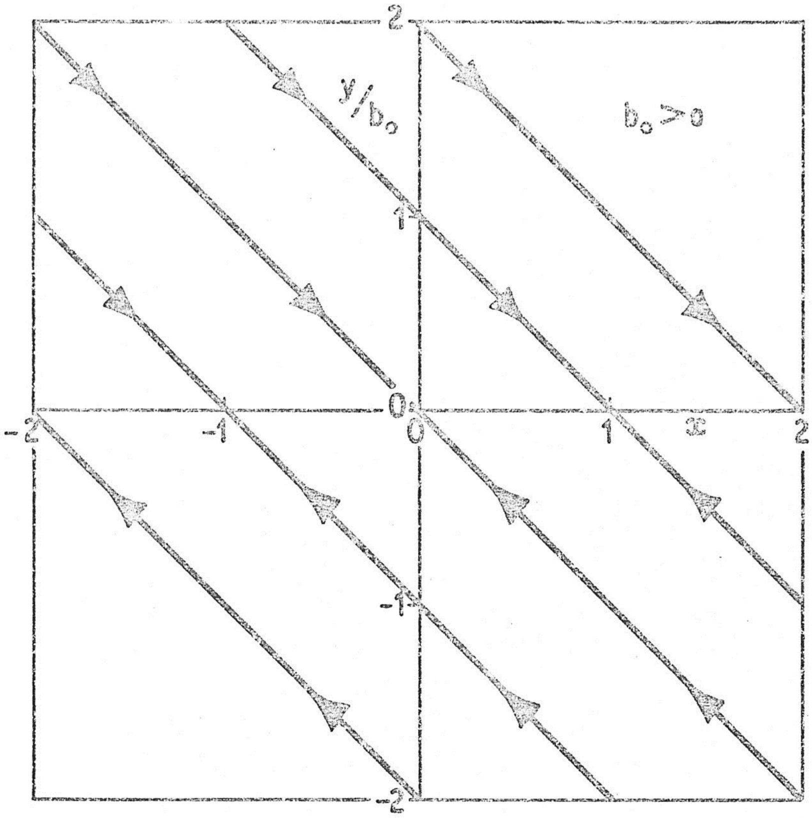


Fig. 1. TRAJECTORIES $b_1 = 0$, $c_1 = 0$, $c_2 = 0$

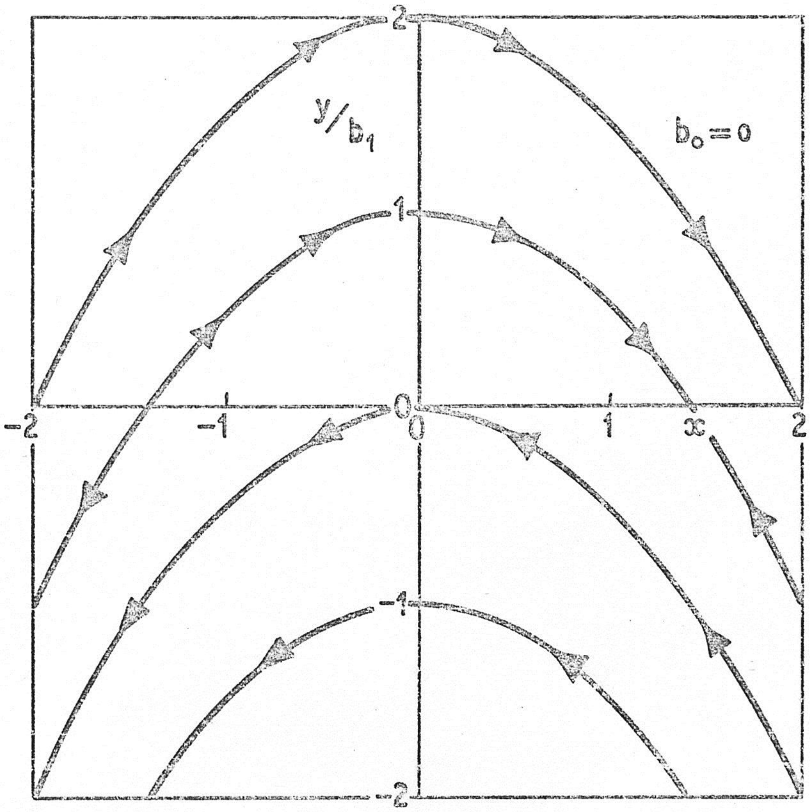


Fig. 2. TRAJECTORIES $b_1 > 0$, $c_1 = 0$, $c_2 = 0$

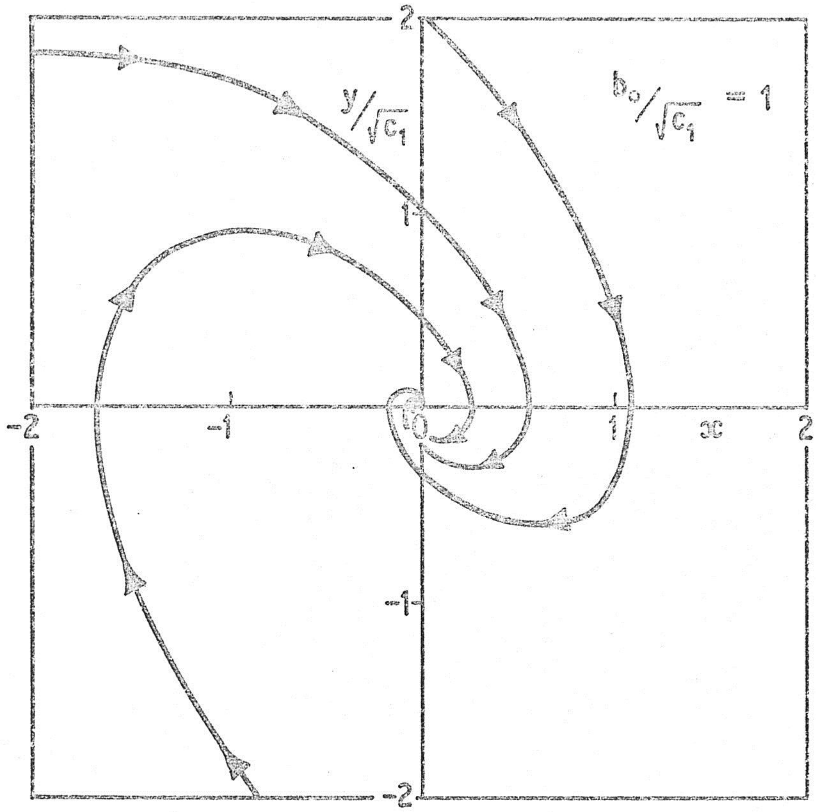


Fig. 3. TRAJECTORIES $b_1=0,\ c_1>0,\ c_2=0$

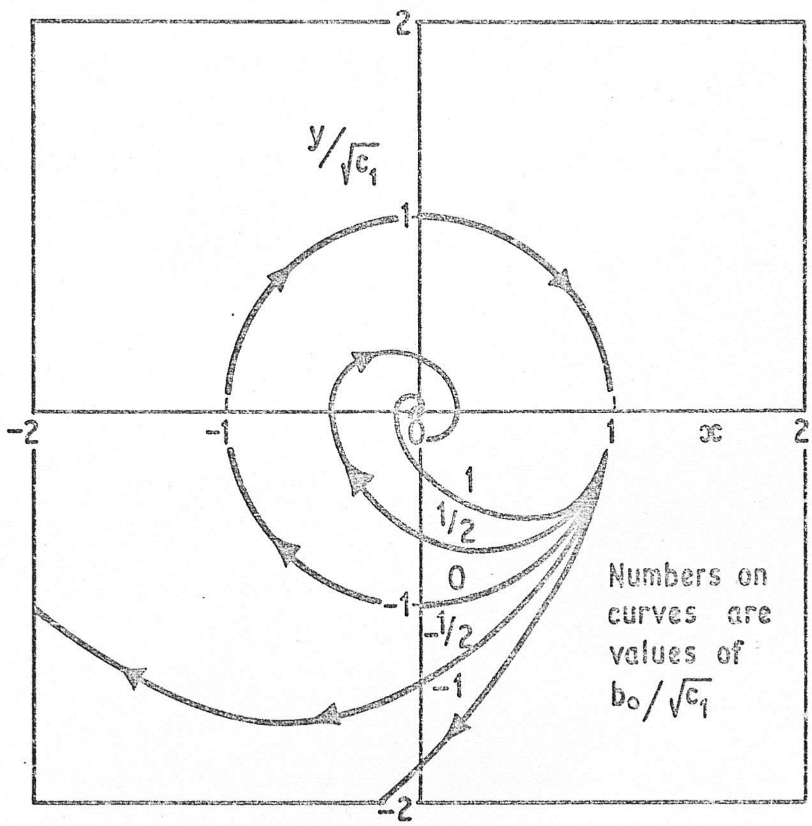


Fig. 4. EFFECT OF DAMPING PARAMETER b_o FOR A LINEAR SYSTEM.

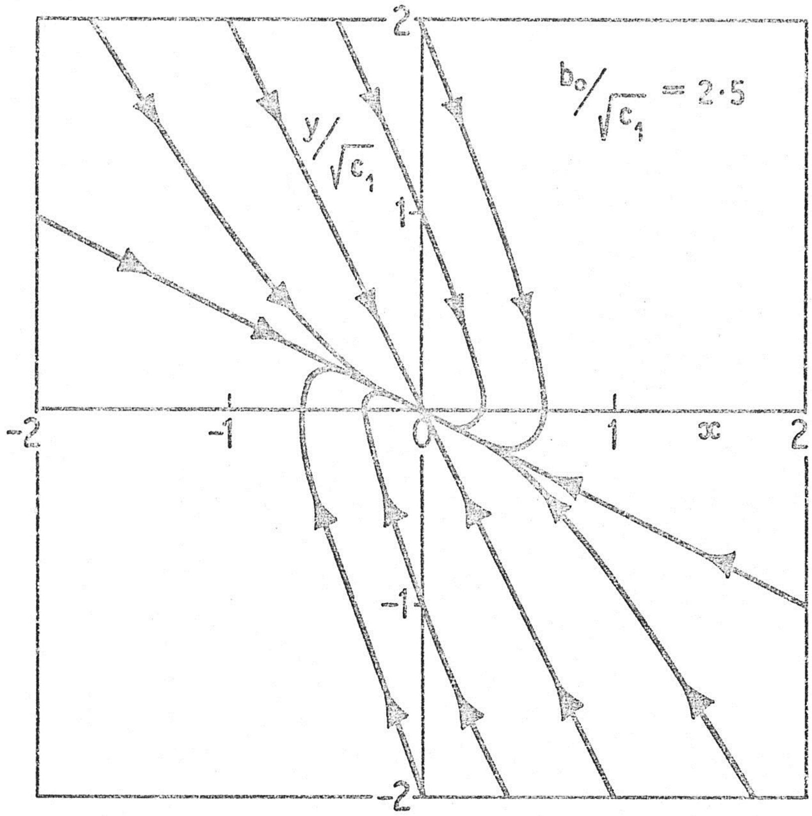


Fig. 5. TRAJECTORIES $b_1 = o$, $c_1 \geq o$, $c_2 = o$

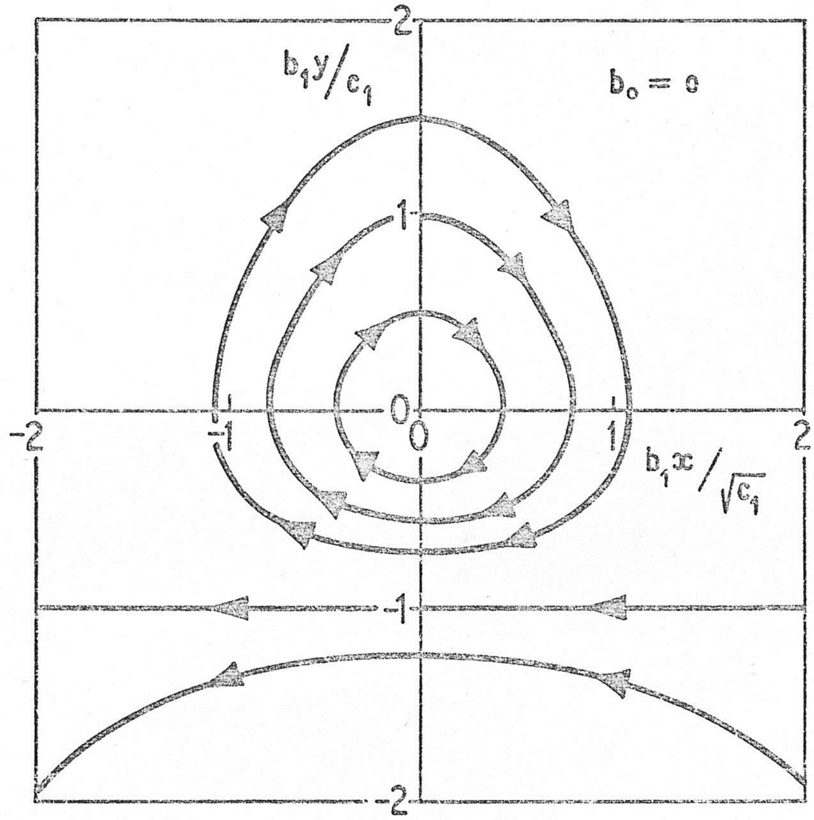
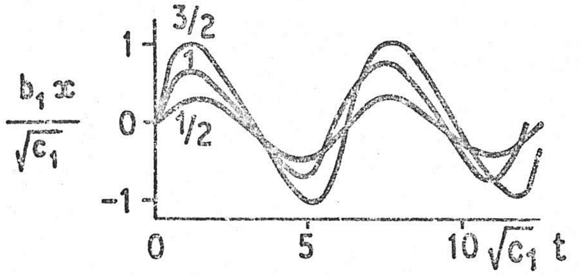


Fig. 6. TRAJECTORIES $b_1 \geq o$, $c_1 \geq o$, $c_2 = o$



Numbers on curves are values of $b_1 y_0/c_1$

Fig. 7. INTEGRAL CURVES $b_0 = 0,\ c_1 > 0,\ c_2 = 0$

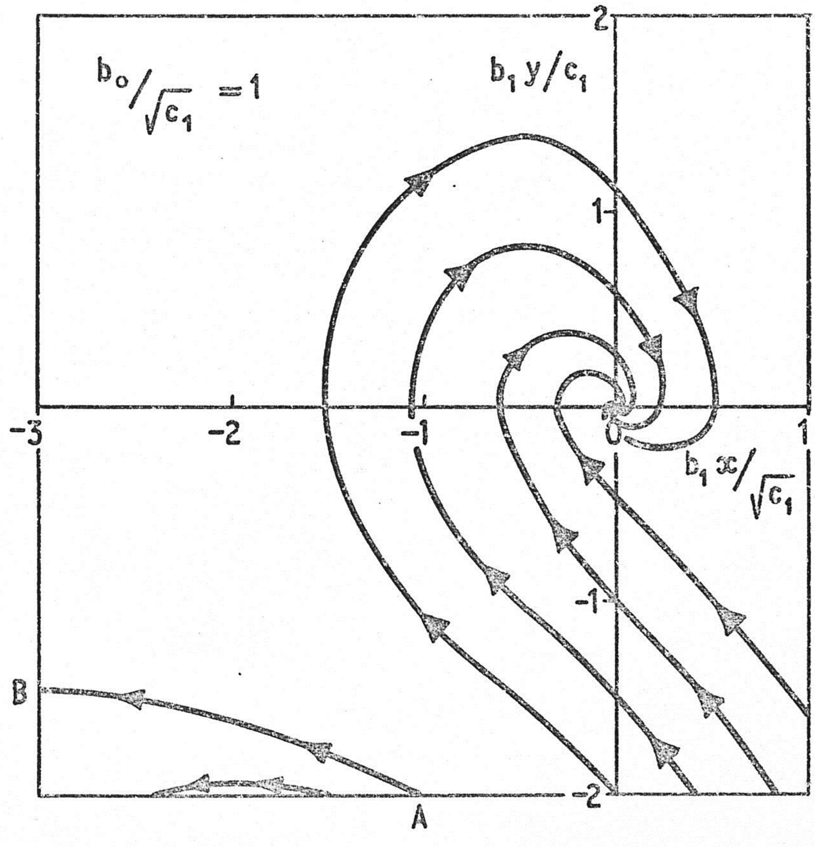


Fig. 8. TRAJECTORIES $b_1 > 0,\ c_1 > 0,\ c_2 = 0$

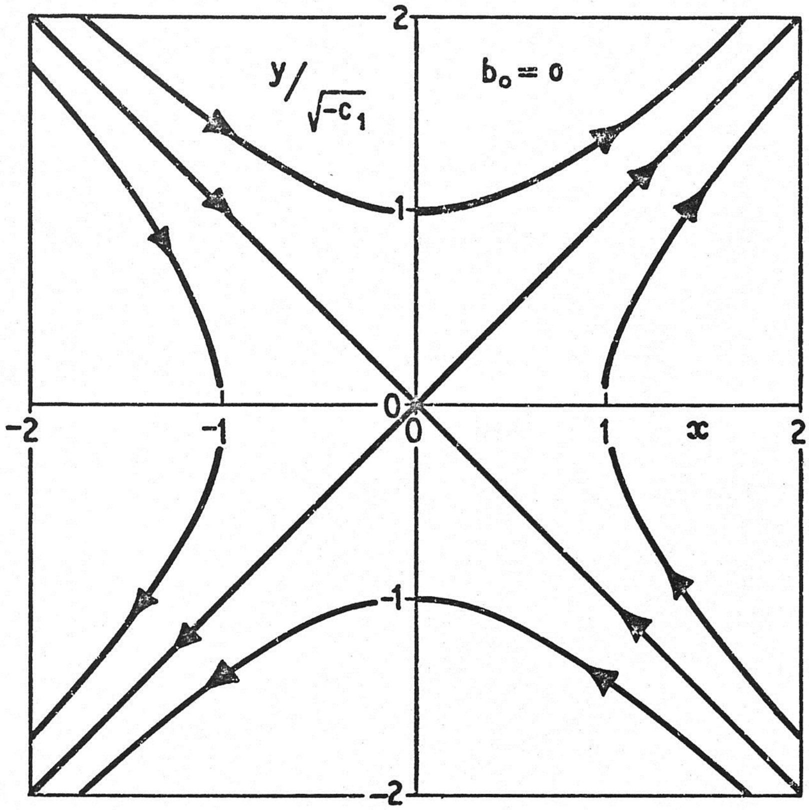


Fig.9. TRAJECTORIES $b_1 = o$, $c_1 < o$, $c_2 = o$

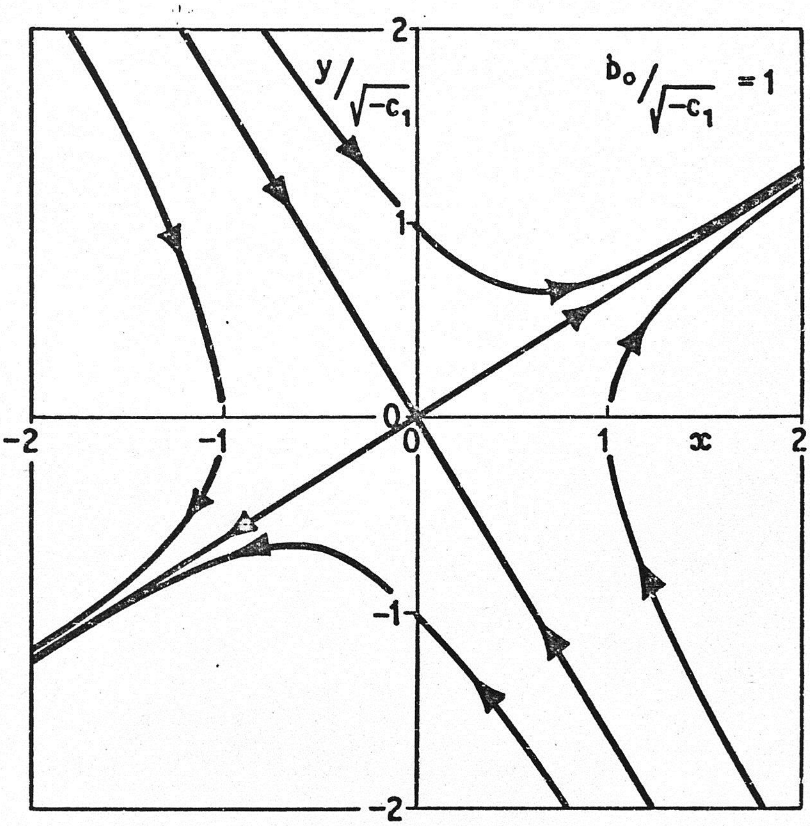


Fig.10. TRAJECTORIES $b_1 = o$, $c_1 < o$, $c_2 = o$

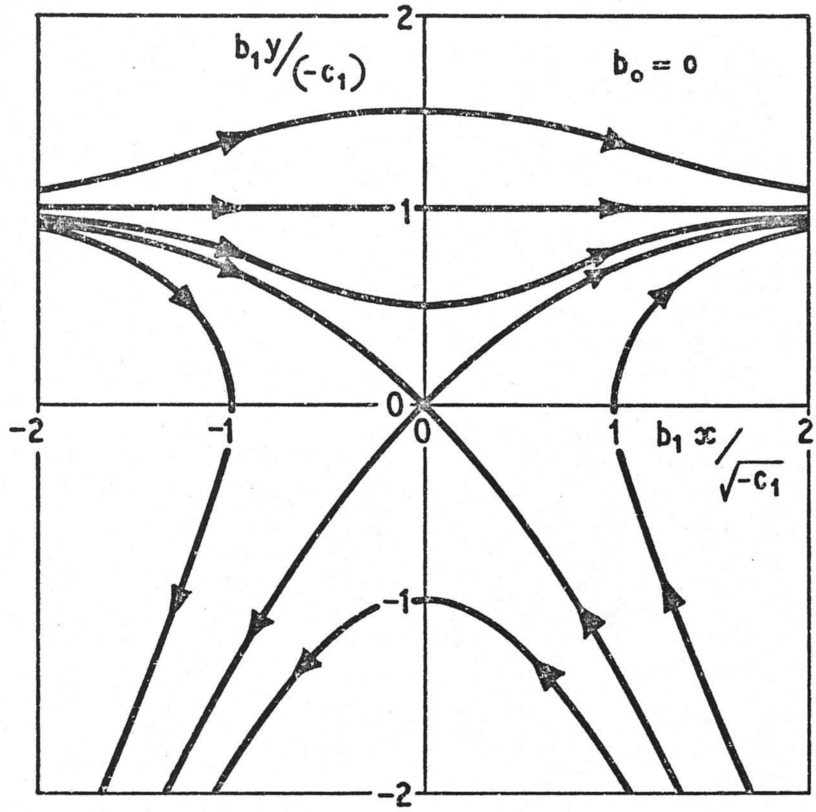


Fig. 11. TRAJECTORIES $b_1 > 0$, $c_1 < 0$, $c_2 = 0$

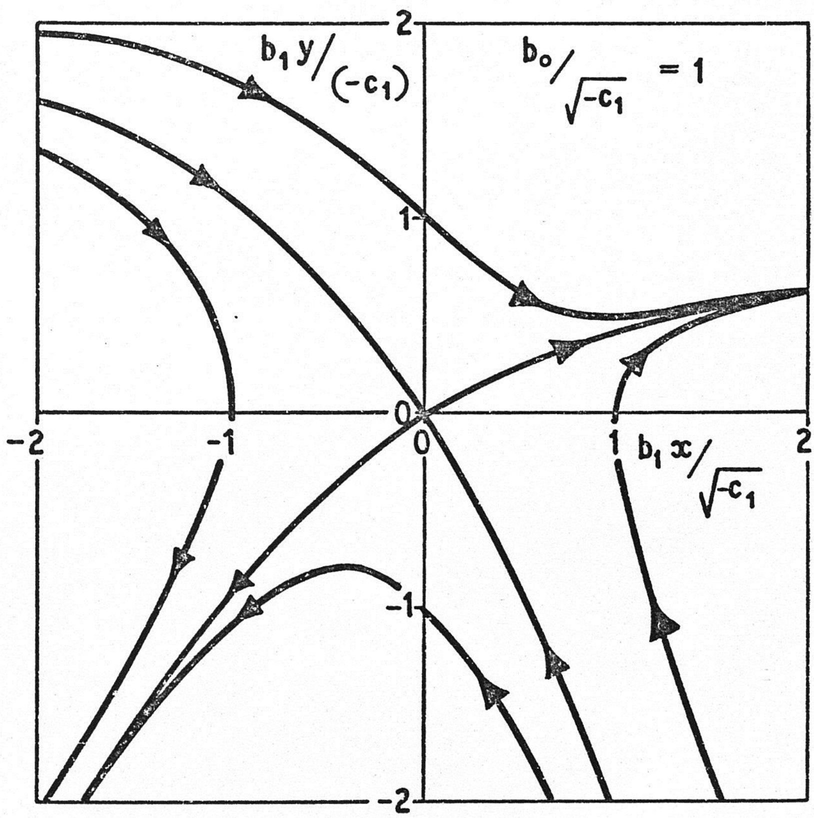


Fig. 12. TRAJECTORIES $b_1 > 0$, $c_1 < 0$, $c_2 = 0$

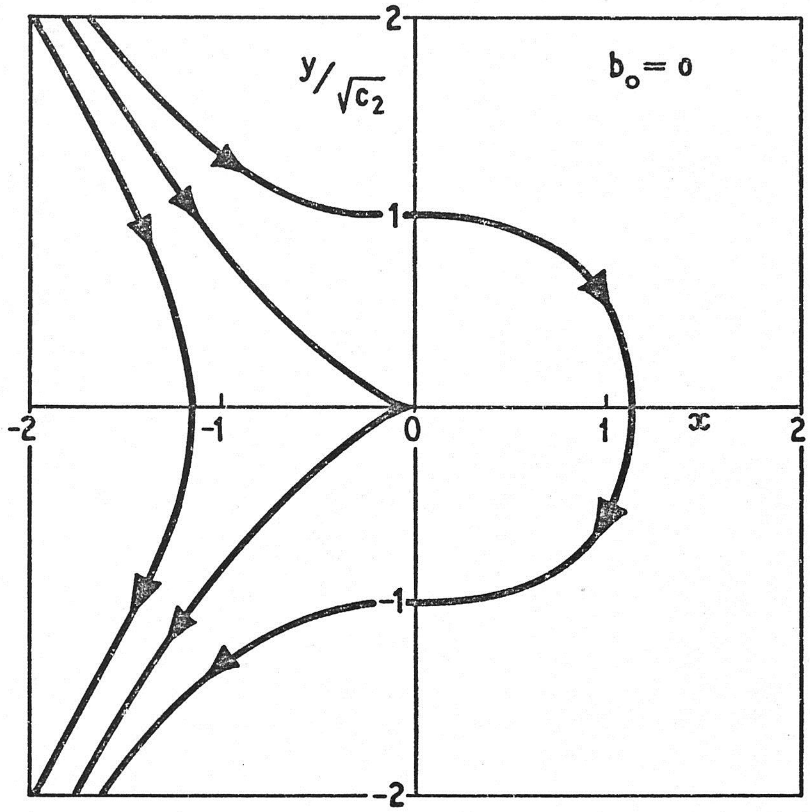


Fig. 13. TRAJECTORIES $b_1 = o$, $c_1 = o$, $c_2 > o$

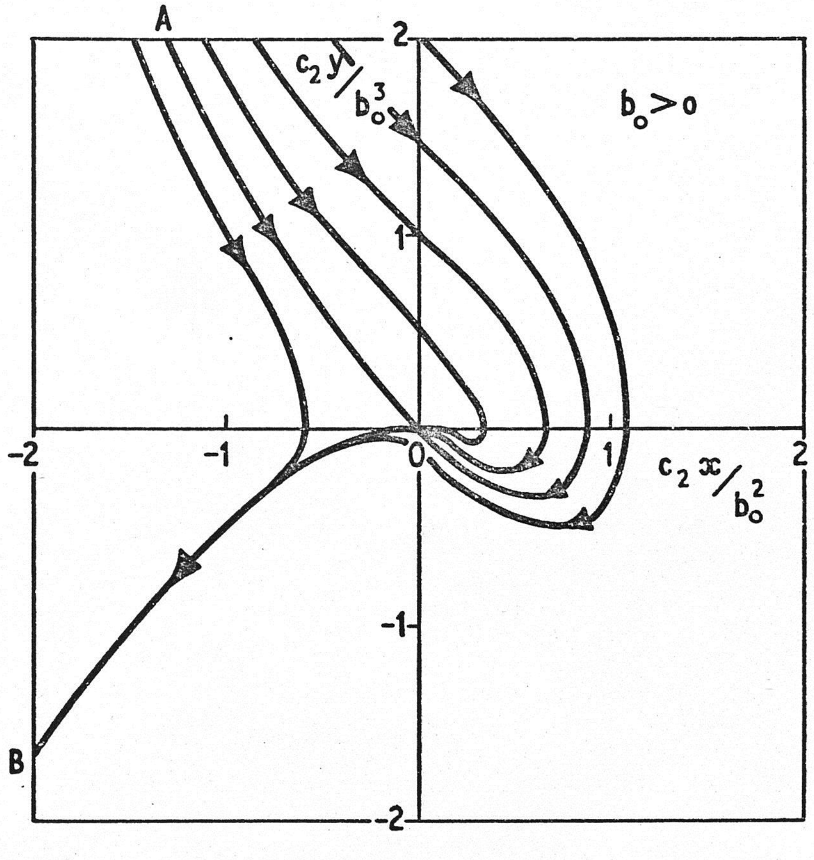


Fig. 14. TRAJECTORIES $b_1 = o$, $c_1 = o$, $c_2 > o$

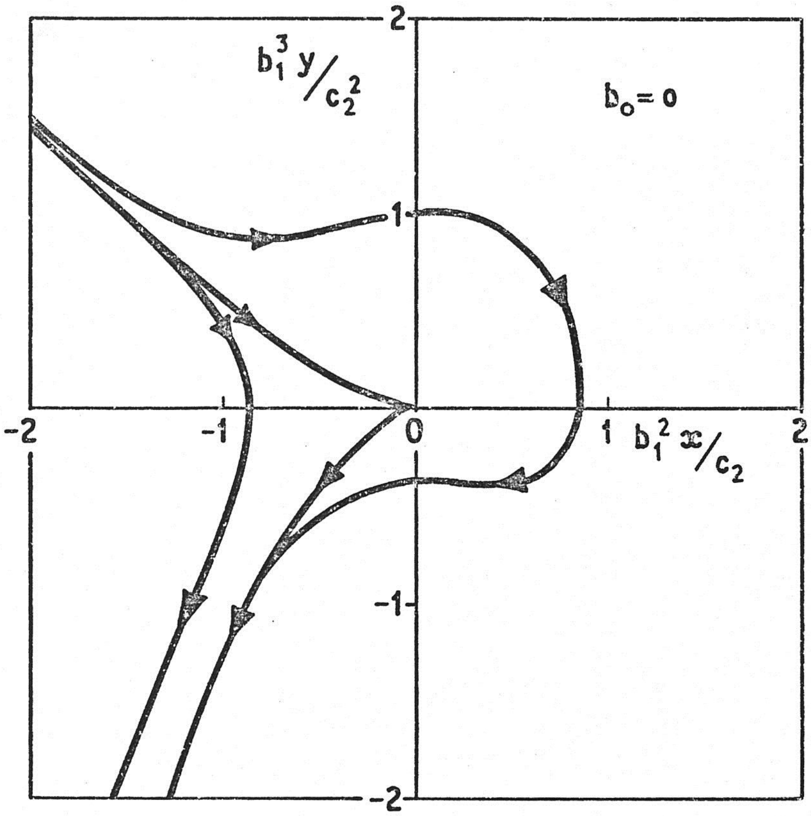


Fig. 15. TRAJECTORIES $b_1 > o$, $c_1 = o$, $c_2 > o$

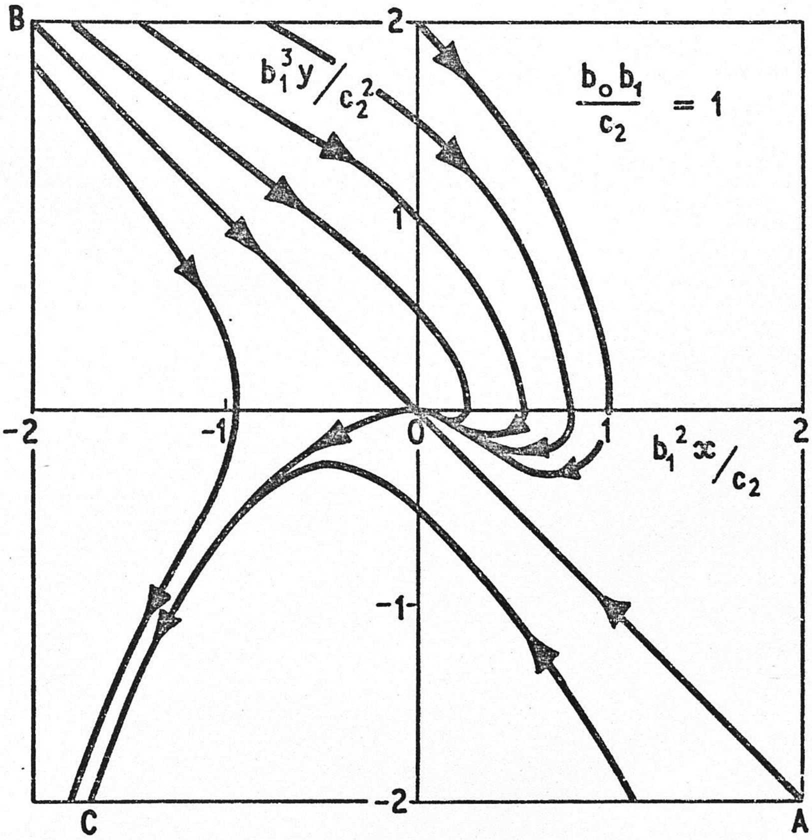


Fig. 16. TRAJECTORIES $b_1 > o$, $c_1 = o$, $c_2 > o$

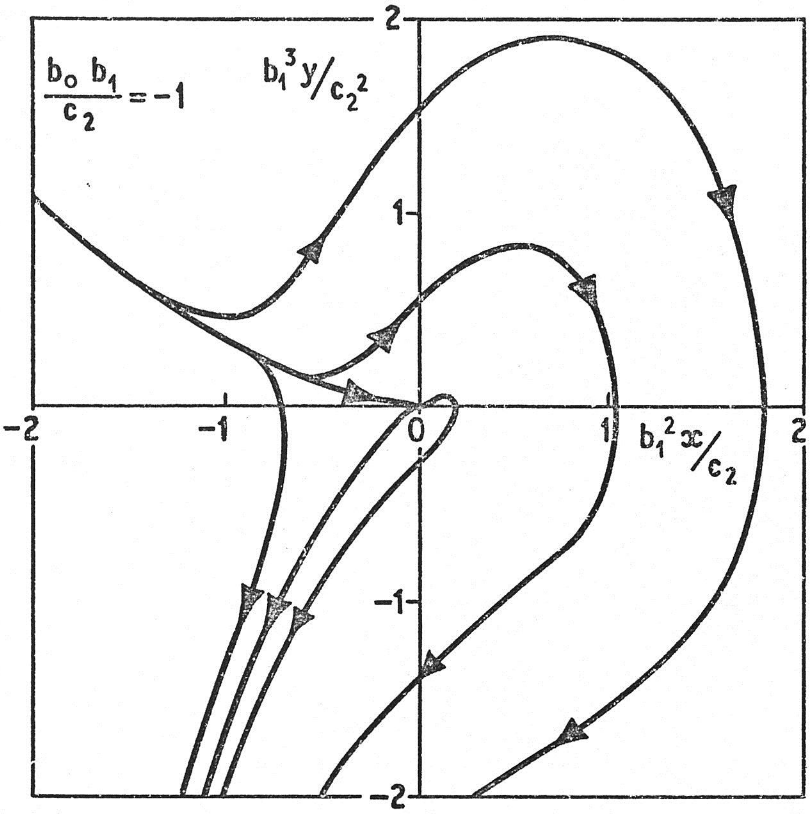


Fig.17. TRAJECTORIES $b_1 > 0$, $c_1 = 0$, $c_2 > 0$

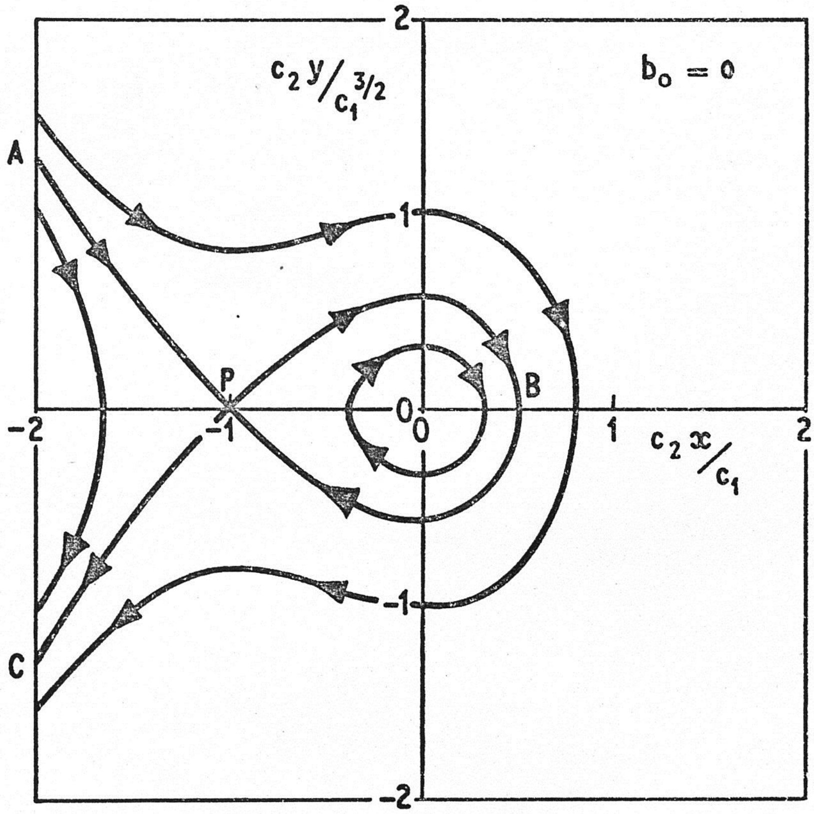
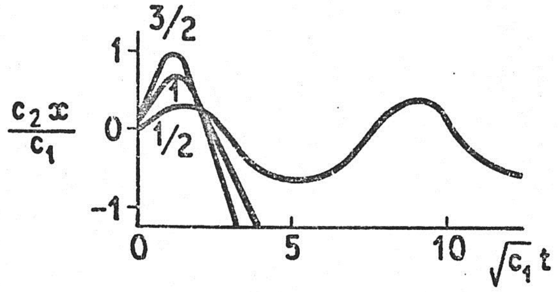


Fig. 18. TRAJECTORIES $b_1 = 0$, $c_1 > 0$, $c_2 > 0$



Numbers on curves are values of $c_2 y_o / c_1^{3/2}$

Fig. 19. INTEGRAL CURVES $b_0 = 0,\ b_1 = 0,\ c_1 > 0$

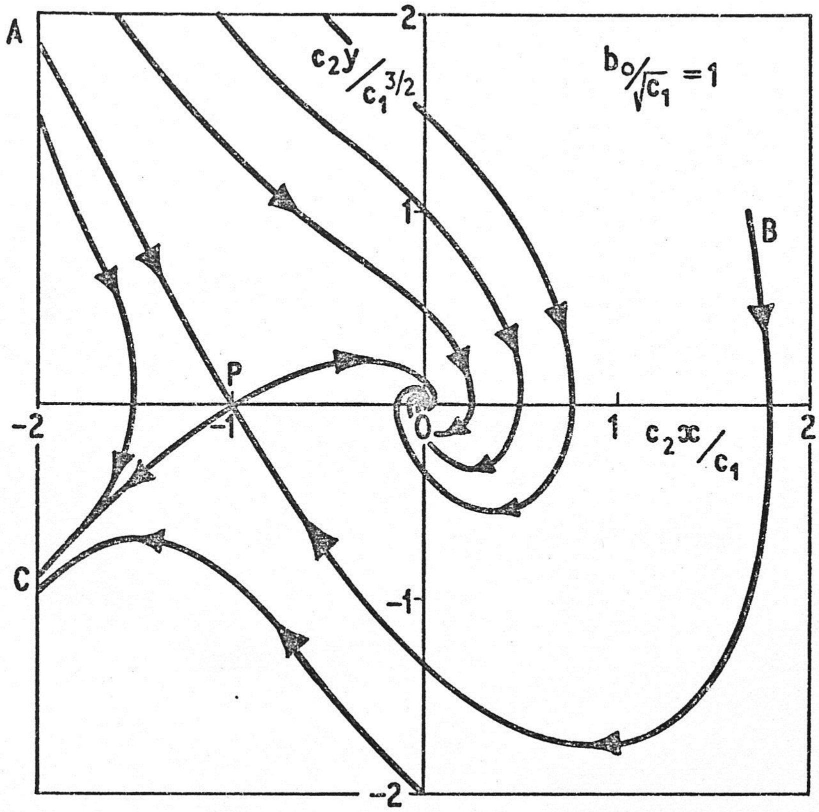


Fig. 20. TRAJECTORIES $b_1 = 0,\ c_1 > 0,\ c_2 > 0$

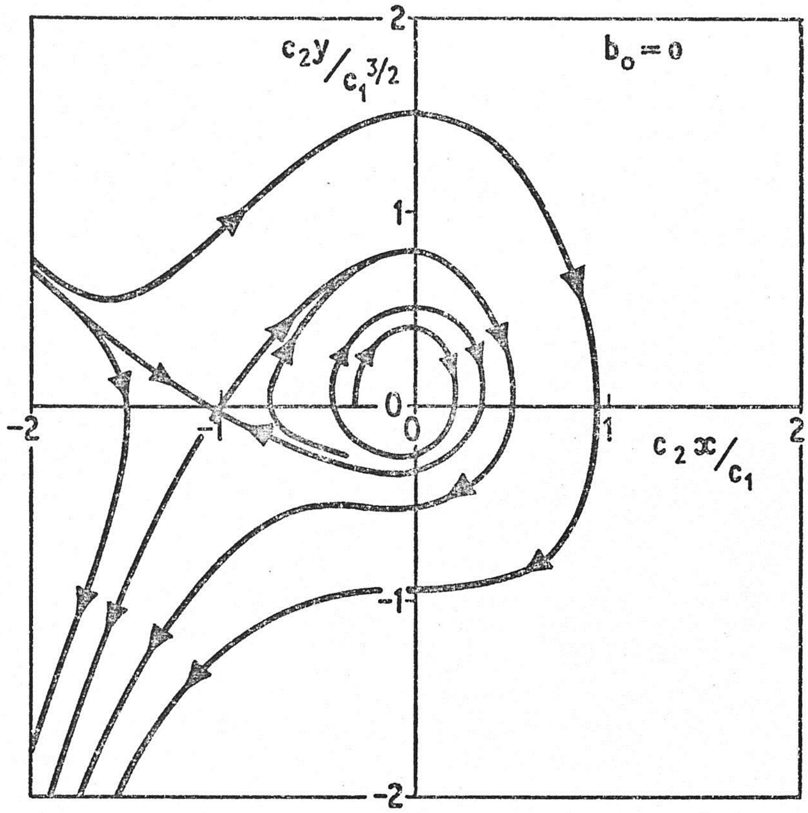


Fig.21. TRAJECTORIES $b_1 > 0$, $c_1 > 0$, $c_2 > 0$

$(b_1 \sqrt{c_1} / c_2 = 1)$

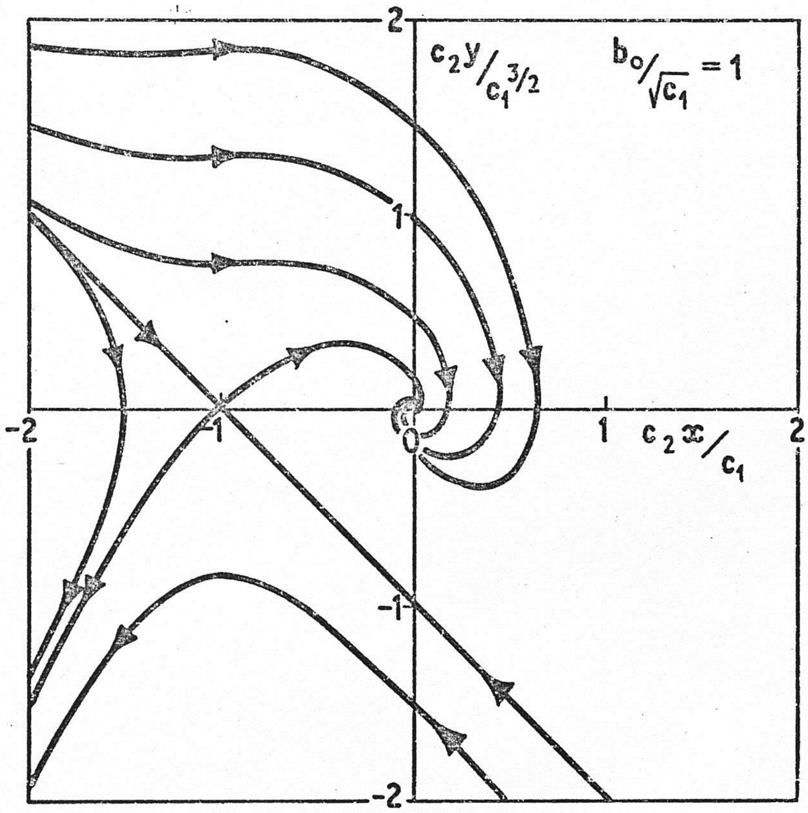


Fig.22. TRAJECTORIES $b_1 > 0$, $c_1 > 0$, $c_2 > 0$

$(b_1 \sqrt{c_1} / c_2 = 1)$

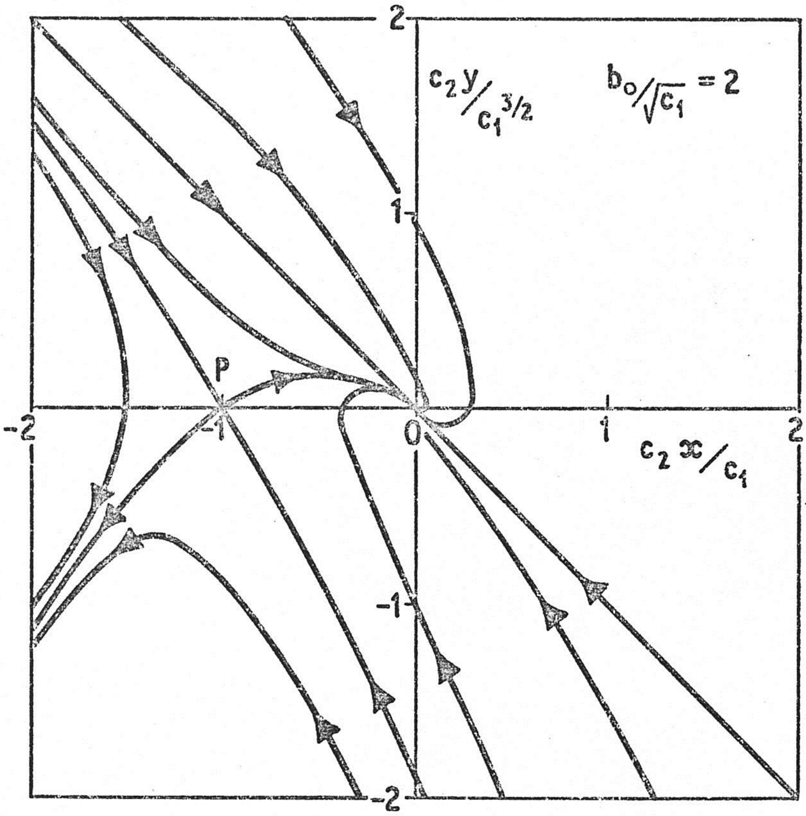


Fig.23. TRAJECTORIES $b_1 > 0$, $c_1 > 0$, $c_2 > 0$

$\left(b_1 \sqrt{c_1} / c_2 = 1\right)$

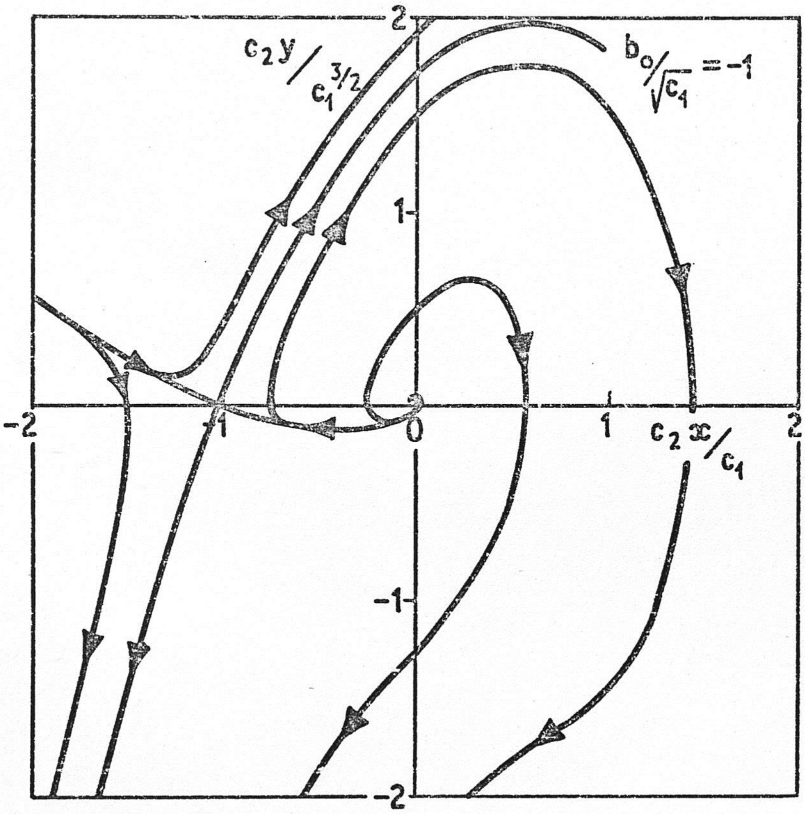


Fig.24. TRAJECTORIES $b_1 > 0$, $c_1 > 0$, $c_2 > 0$

$\left(b_1 \sqrt{c_1} / c_2 = 1\right)$